# INFORMATION TO USERS

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

# TAU FUNCTIONS FOR MATRIX HIERARCHIES

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

by

STANISLAV H. VASILEV

Norman, Oklahoma

1997

UMI Number: 9719904

TAU FUNCTIONS FOR MATRIX HIERARCHIES

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

Approved By

Leonid Dickey

John P Albert

## ACKNOWLEDGEMENTS

I would like to express deep gratitude and appreciation to my thesis advisor Professor Leonid Dickey for his encouragement, helpful suggestions and patience.

I also wish to thank the members of my advisory committee, Professors John Albert, Boris Apanasov, Bruce Mason, and Tomasz Przebinda for their service and encouragement, and Professor Darryl McCullough for his academic advice. I thank Ms. Jane Tarver for her help in preparing this manuscript.

## TABLE OF CONTENTS

**ABSTRACT**

The totality of all zero curvature equations with a rational dependence of connection matrices on a spectral parameter form a hierarchy, which means that all the corresponding vector fields commute. This is the so-called General Zakharov-Shabat (GZS) hierarchy. We consider a subhierarchy of GZS with a given fixed set of poles. The "time variables" depend on three indices, one refers to a chosen pole, the other is a vector index taking values from 1 to $n$ where $n$ is a dimension of the matrices, and the third one corresponds to the order of the pole. In the case of a single pole, the subhierarchy is a generalization of the AKNS hierarchy with matrices of arbitrary dimension and a pole of arbitrary order.

The goal of this work is two-fold. First, we want to construct Grassmannian tau-functions for GZS. We present such a construction for its diagonal tau-functions. Second, we want to give an algebraic-geometrical construction of the Baker and tau functions with a formula connecting them. We have considered the general case when the cross-poles equations are taken into account.

# INTRODUCTION

The equations of zero curvature with rational dependence of the connection matrices on a parameter (or Zakharov-Shabat (ZS) equations) were introduced in [21]. Jimbo and Miwa studied these equations in the context of isomonodromy deformations [17]. The importance of the ZS equations is obvious: many of the nonlinear dynamical systems such as KdV, AKNS, sine-Gordon equations are special cases of ZS. The totality of all ZS equations form a hierarchy [4], which means that all the corresponding vector fields commute. This is the so-called General Zakharov-Shabat (GZS) hierarchy.

We consider a subhierarchy of GZS with a given fixed set of poles. The "time variables" depend on three indices, one refers to a chosen pole, the other is a vector index taking values from 1 to $n$ where $n$ is a dimension of the matrices, and the third one corresponds to the order of the pole. In the case of a single pole, the subhierarchy is a generalization of the AKNS hierarchy with matrices of arbitrary dimension and a pole of arbitrary order.

The goal of this work is two-fold. First, we want to construct Grassmannian tau-functions for GZS. We present such a construction only for the diagonal tau-functions. Second, we want to give an algebraic-geometrical construction of the Baker and tau functions with a formula connecting them. We have considered the general case when the cross-poles equations are taken into account [21].

The notion of a tau function was first suggested for the KP hierarchy by Hirota, Sato et al in [18]. We use their example to explain the notions of a hierarchy and tau function. For that reason we need to recall some definitions and facts:

Let $x$ be a variable and $\partial = \partial/\partial x$. A (formal) pseudodifferential operator in

$x$ is the formal Laurent series in $\partial^{-1}$ with coefficients from $\mathbf{C}[[x]]$.

$$P = \sum_{j=-\infty}^{m} a_j(x)\partial^j, m \in \mathbf{Z}.$$

If $P$ is defined as above and $a_m \neq 0$, then $m$ is called the order of $P$. One can add and multiply such series. The multiplication of two pseudodifferential operators is defined according to the commutation rule:

$$\partial^k f = f\partial^k + \binom{k}{1} f'\partial^{k-1} + ..., \ k \in \mathbf{Z}, \ \binom{k}{i} = \frac{k(k-1)..(k-i+1)}{i!}.$$

Now let

$$L = \partial + u_0\partial^{-1} + u_1\partial^{-2} + .. .$$

The linear space of all operators $L$ of this form will be our phase space $\mathcal{L}$. Let $B_k = L_+^k$ be the positive (differential) part of $L^k$. Then the KP hierarchy is defined by the following differential equations for $L$:

$$\partial_k L = [B_k, L], \ \partial_k = \frac{\partial}{\partial x_k},$$

where $x_k$, $k = 1, ..$ are some variables, $x_1 = x$, and $[B_k, L]$ is the commutator of the two operators.

**Remark.** The KP equations define vector fields on the phase space $\mathcal{L}$. We need to check that for any $L \in \mathcal{L}$ the operator $\partial_k L = [B_k, L]$ lies in the tangent space $T_L\mathcal{L} \cong \mathcal{L}$. Really, the vector field can be rewritten as

$$[B_k, L] = [L_+^k, L] = [L^k - L_-^k, L] = -[L_-^k, L],$$

from where it is clear that this is an operator of order less or equal than $-1$; therefore it belongs to $\mathcal{L}$.

Often the KP differential equations are written in the form of "zero curvature equations":

$$\partial_m B_n - \partial_n B_m = [B_m, B_n].$$

If we take $n=2,\ m=3,\ x_2=y,\ x_3=t,\ 2u_0=u$, we obtain the KP equation (which gives the name to the whole hierarchy):

$$3u_{yy}-(4u_t-u_{xxx}-6uu_x)_x=0.$$

It can be shown that the above vector fields $\partial_k$ commute

$$(\partial_m\partial_n-\partial_n\partial_m)L=0,$$

which means that the KP equations indeed form a hierarchy.

We will be looking for solutions to the KP hierarchy in the form:

$$L=\Phi\partial\Phi^{-1},\ \Phi=1+\sum_0^\infty w_i\partial^{-i-1}.$$

The series $\Phi$ is formal. The above representation is called "dressing" of the operator $\partial$.

Pseudodifferential operators do not act on functions (unless their negative parts are zero, in which case they are differential operators). Now we define their action on exponents. Let $\xi(x,z)=\sum_1^\infty x_iz^i$, then we can define the action of $\partial^k,\ k\in\mathbf{Z}$ on $\exp\xi(x,z)$:

$$\partial^k\exp\xi(x,z)=z^k\exp\xi(x,z).$$

The above rules can be extended to any pseudodifferential operator:

$$\sum_i a_i\partial^i\exp\xi(x,z)=\sum_i a_iz^i\exp\xi(x,z).$$

Now we can define the Baker function (also known as wave function) of the hierarchy:

$$w=\Phi\exp\xi(x,z)=(1+w_0z^{-1}+w_1z^{-2}+..)\exp\xi(x,z)=\hat{w}(x,z)\exp\xi(x,z),$$

where $\Phi$ is the dressing operator of $\partial$. The function $w$ is determined up to a multiplier, which is a series in $z^{-1}$ constant in $x$.

The Baker function satisfies the following linear equations:

$$Lw = zw,$$

$$\partial_m w = B_m w, \ (B_m = L_+^m), \ m = 1, 2, .. \, .$$

The above system of equations has a solution $w$ if and only if the operator $L$ satisfies the KP hierarchy.

If $w = \hat{w} \exp \xi(x, z)$ is a Baker function for the KP hierarchy, then a function $\tau(x_1, x_2, ...)$ exists such that

$$\hat{w}(x, z) = \frac{\tau\left(x_1 - \frac{1}{z}, x_2 - \frac{1}{2z^2}, x_3 - \frac{1}{3z^3}, ...\right)}{\tau(x_1, x_2, ...)}.$$

The above relation is the famous Sato's formula. The function $\tau$ is called tau-function of the KP hierarchy corresponding to the Baker function $w$.

The tau function provides a solution to the hierarchy. Really, if we know $\tau(x)$, using the Sato formula we obtain the Baker function $w(x, z)$, next from $w(x, z)$ we can recover the dressing operator $\Phi$, and finally we obtain a solution $L$ to KP by "dressing" the operator $\partial$: $L = \Phi \partial \Phi^{-1}$. The significance of the tau function lies in the fact that a single function (not infinitely many!) gives a solution to the hierarchy. (Recall that $L = \partial + u_0 \partial^{-1} + u_1 \partial^{-2} + ..$, which means that in order to have a solution $L$ we need to know all the functions $u_0$, $u_1, ...$ .)

There are two very important examples of tau functions: the soliton and the algebraic-geometrical cases. We will now recall the soliton example.

Let $\alpha_k, \beta_k, a_k, k = 1, .., N$ be distinct complex numbers. ($N$ is the soliton number.) Define the functions

$$y_k = \exp \xi(x, \alpha_k) + a_k \exp \xi(x, \beta_k), \ k = 1, .., N,$$

where the function $\xi(x, z)$ is defined as before. The functions $y_k$ satisfy the equations:

$$\partial_m y_k = \partial^m y_k, \ k = 1, .., N; \ m = 1, 2, .. \, .$$

**Remark.** The property above is central to the soliton example. Actually if we take any set of functions $y_k$ that satisfy these equations, we can construct a solution to the hierarchy.

The corresponding dressing operator looks like

$$\Phi = \frac{1}{\Delta}\begin{vmatrix} y_1 & \dots & y_N & \partial^{-N} \\ y_1' & \dots & y_N' & \partial^{-N+1} \\ \dots & \dots & \dots & \dots \\ y_1^{(N)} & \dots & y_N^{(N)} & 1 \end{vmatrix},$$

where $\Delta$ is the Wronskian of the functions $y_1, .., y_N$. Then the operator $L = \Phi\partial\Phi^{-1}$ satisfies the KP hierarchy. The Baker function is

$$\begin{aligned} w =& \Phi \exp \xi(x,z) \\ =& \frac{1}{\Delta}\begin{vmatrix} y_1 & \dots & y_N & z^{-N} \\ y_1' & \dots & y_N' & z^{-N+1} \\ \dots & \dots & \dots & \dots \\ y_1^{(N)} & \dots & y_N^{(N)} & 1 \end{vmatrix} \exp \xi(x,z). \end{aligned}$$

The tau function for the $N$-soliton solution is

$$\tau(x_1, x_2, \dots) = \Delta = \begin{vmatrix} y_1 & y_2 & \dots & y_N \\ y_1' & y_2' & \dots & y_N' \\ \dots & \dots & \dots & \dots \\ y_1^{(N-1)} & y_2^{(N-1)} & \dots & y_N^{(N-1)} \end{vmatrix}.$$

In order to check that this function is a tau function we need to introduce the translation operator G(z):

$$G(z)f(x) = f(x_1 - \frac{1}{z}, x_2 - \frac{1}{2z^2}, x_3 - \frac{1}{3z^3}, \dots).$$

Then we can obtain

$$G(z)y_k = y_k - \frac{1}{z}y_k'.$$

Thus the right hand side of Sato's formula will look like

$$\frac{G(z)\tau}{\tau} = \begin{vmatrix} y_1 - \frac{1}{z}y_1' & \dots & y_N - \frac{1}{z}y_N' \\ y_1' - \frac{1}{z}y_1'' & \dots & y_N' - \frac{1}{z}y_N'' \\ \dots & \dots & \dots \\ y_1^{(N-1)} - \frac{1}{z}y_1^{(N)} & \dots & y_N^{(N-1)} - \frac{1}{z}y_N^{(N)} \end{vmatrix}.$$

The determinant in the formula for the Baker function can be reduced to this form if we divide its second row by $z$ and subtract it from the first one, divide the third row by $z$ and subtract it from the second one etc. Thus the function $\tau(x) = \Delta$ is a tau function for the hierarchy corresponding to the $N$-soliton solution.

The soliton example is important for two reasons. First, it provides the hierarchy with a simple and elegant solution. Second, it gives a hint how to find other solutions. The general idea is that the tau function is a determinant of some matrix [18]. That idea was realized for KP [19], where a Grassmannian definition of the tau function was given. For each element of the Grassmannian manifold there is an associated with it mapping, whose determinant is a tau function for the hierarchy. The soliton and the algebraic-geometrical examples can be obtained as special cases if the corresponding elements of the Grassmannian are properly chosen.

The KP is a scalar hierarchy generated by a pseudodifferential operator. AKNS is a hierarchy of another type – generated by a matrix first order differential operator, which depends linearly on a spectral parameter:

$$L = \partial + U - zA, \ \partial = \partial_x,$$

$$A = \mathrm{diag}(a_1, ..., a_n), \ U = (u_{ij}), \ i, j = 1, .., n.$$

A Grassmannian construction of the algebraic-geometrical solution to AKNS was given in [5]. The corresponding tau function in this case is a Riemann theta function (as for KP). Based on the analysis of the soliton solution, a general Grassmannian definition of a tau function for AKNS was obtained in [9].

The next step is to consider matrix hierarchies generated by linear operators depending rationally on the spectral parameter. These are ZS hierarchies. In section 1.1 we recall the definitions of the GZS hierarchy, its Baker and tau functions. Our main result in chapter 1 is formulated in Theorem 1.4.1, where we provide a formula for the diagonal elements of a general Grassmannian tau function along with a formula expressing the diagonal Baker function in terms of the tau functions.

In chapter 2 we present an algebraic-geometrical construction of the Baker and the tau functions for ZS. In section 2.2 we construct an algebraic-geometrical element of the Grassmannian. A formula for the Baker-Akhiezer function is given in section 2.3. Finally, the main result of that chapter is formulated in section 2.4 (Theorem 2.4.1). We give a formula for the algebraic-geometrical tau function along with a Sato-like formula for the Baker-Akhiezer function.

# Chapter I

# DIAGONAL TAU FUNCTIONS FOR THE HIERARCHY GZS

## 1.1. Definition of the ZS hierarchy

Here we use the definitions and notation from [4]. In order to keep the text self-contained, we need to recall some of them. (For more detail and proofs see [4]). Let $a_k,\ k = 1, ..., m$ be a fixed set of complex numbers. For every $k$ we define

$$\hat{w}_k = \sum_0^\infty w_{ki}(z - a_k)^i$$

as a formal series. The coefficients $w_{ki}$ are $n \times n$ matrices, whose entries are taken as generators of a differential algebra. Let us define the resolvents:

$$R_{k\alpha} = \hat{w}_k E_\alpha \hat{w}_k^{-1}; \quad R_{k\alpha l} = R_{k\alpha}(z - a_k)^{-l},$$

where $E_\alpha$ is the $n \times n$ matrix whose only non-zero entry $(E_\alpha)_{\alpha\alpha}$ is equal to 1.

The following equations define a ZS hierarchy corresponding to the fixed set $\{a_k\}$:

$$\partial_{k\alpha l}\hat{w}_{k_1} = \begin{cases} -R^+_{k\alpha l}\hat{w}_{k_1} & \text{if } k = k_1 \\ R^-_{k\alpha l}\hat{w}_{k_1} & \text{otherwise,} \end{cases}$$

where $t_{k\alpha l}$ are some variables, $\partial_{k\alpha l} = \partial/\partial t_{k\alpha l}$, and $R^+_{k\alpha l}$, $R^-_{k\alpha l}$ are correspondingly the positive and negative parts of the series $R_{k\alpha l}$. It can be shown that all operators $\partial_{k\alpha l}$ commute.

Let

$$w_k = \hat{w}_k \exp \xi_k \quad \text{where} \quad \xi_k = \sum_{l=0}^{\infty} \sum_{\alpha=1}^{n} t_{k\alpha l} E_\alpha (z - a_k)^{-l}.$$

The collection $w = \{w_k\}$ is called the formal Baker function of the hierarchy.

A dressing formula holds:

$$w_{k_1} \partial_{k\alpha l} w_{k_1}^{-1} = \partial_{k\alpha l} - B_{k\alpha l}, \quad B_{k\alpha l} = R_{k\alpha l}^{-}$$

All the operators $\partial_{k\alpha l} - B_{k\alpha l}$ commute. Now we consider linear combinations of these operators:

$$L = \sum_{k\alpha l} \lambda_{k\alpha l} (\partial_{k\alpha l} - B_{k\alpha l}) = \partial + U,$$

where $\partial = \sum_{k\alpha l} \lambda_{k\alpha l} \partial_{k\alpha l}$ and $U = \sum_{k\alpha l} \lambda_{k\alpha l} B_{k\alpha l}$. The commutativity of two such operators yields Zakharov-Shabat type equations:

$$\partial U_1 - \partial_1 U = [U_1, U].$$

The functions $U$ and $U_1$ depend rationally on $z$ and vanish at infinity. This is a special case of ZS equations. The general case can be obtained with a gauge transformation of the Baker function.

**Remark.** The ZS hierarchy admits a group of similarity transformations, allowing us to neglect the dependence on $t_{k\alpha 0}$. Really, from the equations of the hierarchy we can obtain $\partial_{k\alpha 0} \hat{w}_{k_1} = 0, \;\; k_1 \neq k$ and $\partial_{k\alpha 0} \hat{w}_k = -E_\alpha \hat{w}_k$. Therefore the dependence of $\hat{w}_k$ on the times $t_{k_1 \alpha 0}$ looks like

$$\hat{w}_k = \exp\left( - \sum_{\alpha=1}^{n} t_{k\alpha 0} E_\alpha \right) \hat{w}_k(0).$$

From now on we will neglect the times $t_{k\alpha 0}$; we know how to restore that dependence.

Let $C_k$ be disjoint circles on the Riemann sphere $\mathbf{CP}^1$ around the fixed points $a_k$, $k = 1, ..., m$, and $\Omega$ be the part of $\mathbf{CP}^1$ outside these circles. Let $H_k$ be the

Hilbert spaces of vector functions $\mathbf{f}(z) \in \mathbf{C}^n$ on the circles. Subspaces $H_k^+$ and $H_k^-$ consist of functions, that can be correspondingly expanded in non-negative and negative powers of $z - a_k$. Let $H = \oplus_k H_k$ and $H^+$ consists of $\{\mathbf{f}_k\}$ such that $\mathbf{f}_k \in H_k^+$ and $\sum_k \mathbf{f}_k(a_k) = 0$. Now, let $H^*$ consist of $\mathbf{f} = \{\mathbf{f}_k\} \in H$ such that $\mathbf{f}_k$ are boundary values on the circles $C_k$ of a vector-function holomorphic in the domain $\Omega$. It can be shown that $H = H^* \oplus H^+$. We will also need the space $H^- = \oplus_k H_k^-$ and the natural projector $P^- : H \to H^-$. Let $P^*$ be the projector $P^* : H \to H^*$. Its action is given by the formula:

$$P^*(\{\mathbf{f}_k\}) = \sum_i \mathbf{f}_i^- + \mathbf{c},$$

where $\mathbf{c} = m^{-1} \sum_k \mathbf{g}_k(a_k)$, $\mathbf{g}_k = \mathbf{f}_k^+ - \sum_{i \neq k} \mathbf{f}_i^-$.

An element of the Grassmannian, $W \in \mathbf{Gr}$, is a subspace of $H$ with the following properties:

i) the projector $P^* : H \to H^*$ restricted to $W$ is a bijection,

ii) $(z - a_1)^{-1} W = (z - a_2)^{-1} W = ... = (z - a_k)^{-1} W \subset W$.

An example of an element $W$ of the Grassmannian will be given in chapter II. We will construct there $W$ for the algebraic-geometrical case and give formulas for the corresponding Baker and tau functions. Another example (for the soliton case) can be found in [4].

We consider vectors here as row vectors. We say that a matrix belongs to $W$ if all its rows do. Now let us consider the following transformation of the Grassmannian. If $\mathbf{f} = \{\mathbf{f}_k\} \in W$ then $\mathbf{f} \exp \xi = \{\mathbf{f}_k \exp \xi_k\}$, where $\xi_k$ is the same as above. The set of all $\mathbf{f} \exp \xi$ we call $W \exp \xi$. The subspace $W \exp \xi$ will belong to the Grassmannian $\mathbf{Gr}$ for almost all $t_{k\alpha l}$.

A Grassmannian Baker function, corresponding to an element $W$ of the Grassmannian, is a matrix-function $w \in W$ such that:

i) $P^* w \exp(-\xi) = c$, where $c$ is constant in $z$ but can depend on variables $t_{k\alpha l}$,

ii) $\partial_{k\alpha l} w_i \in (z-a_j)^{-1}W$ for every $(k\alpha l)$ (the subspace on the right-hand side does not depend on $j$, see the definition of the Grassmannian).

Let us introduce the translation operator $G_{k\beta}(z)$

$$G_{k\beta}(z)f(t) = f(..,t_{q\gamma l} - \delta_{kq}\delta_{\beta\gamma} l^{-1}(z-a_k)^l,..)$$

A tau-function for the ZS hierarchy should satisfy a relation of the type

$$\hat{w}_{k,\alpha\beta}(t,z) = \frac{G_{k\beta}(z)\tau_{k,\alpha\beta}(t)}{\tau(t)} p_{k\alpha\beta}(z), \qquad 1.1.1$$

where $w = \{w_k\}$ is the Baker function, $w_k = \hat{w}_k e^{\xi_k}$, and $p_{k\alpha\beta}(z)$ is a function holomorphic in a neighborhood of $a_k$ and it does not depend on $t$. In section 1.4 (see Theorem 1.4.1.) we find such a tau function and the corresponding relation for the diagonal case $\beta = \alpha$. In section 2.4 (Theorem 2.4.1) we construct a tau function for the algebraic-geometrical solution and give along a formula like (1.1.1).

## 1.2. Tau functions for the single pole hierarchy

In [19] Segal and Wilson developed Sato's ideas [18] and gave a Grassmannian construction for tau functions to the KP hierarchy. For each element of the Grassmannian they defined the corresponding tau function as a determinant of some mapping. Dickey [8] modified the Segal-Wilson definition of the tau function. Analyzing the soliton example he gave a Grassmannian definition for the tau functions to AKNS, which is a special case of ZS hierarchy ( single pole at infinity).

Let us recall the construction in [8]. First we adopt a notation more convenient to compare with the general case of many poles: the pole is at a finite point (say 0), not at infinity.

The Hilbert space $H$ consists of vector functions on the unit circle. $H^+$ is subspace of all series strictly positive powers and $H^-$ is the subspace of all non-positive powers. (In the case of a single pole $H^* = H^-$.) Clearly $H = H^+ \oplus H^-$. Let $W$ be an element of the Grassmannian and $w$ be the corresponding normalized Baker function. Let $\mathbf{g}'_\beta = \mathbf{I} - \zeta/z\mathbf{E}_\beta$ and $\mathbf{g} = e^\xi$, where

$$\xi = \sum_{\gamma=1}^{n} \sum_{l=1}^{\infty} t_{\gamma l} z^{-l} \mathbf{E}_\gamma.$$

We need a basis in $H^+$. An obvious choice is:

$$\epsilon_{l\gamma} = z^l \mathbf{e}_\gamma, \ l = 1, 2, \dots \ ; \ \gamma = 1, .., n.$$

The vector $\mathbf{e}_\gamma$ is row-vector of length $n$, whose only non-zero entry is 1 at the $\gamma$-th position. Let us introduce a linear operator $R_{\alpha\beta} : H^+ \to H$ ($\beta \neq \alpha$):

$$R_{\alpha\beta} : \begin{cases} \epsilon_{\beta 1} \mapsto -\mathbf{e}_\alpha & \\ \epsilon_{\gamma l} \mapsto \epsilon_{\gamma l} & \text{for all other indices.} \end{cases}$$

Now we need to define some mappings. Let $l_W : H \to H^+$ be the projection parallel to $W$, $T_W(\mathbf{g}) = l_W \circ \mathbf{g} : H^+ \to H^+$, and $T_{W\alpha\beta}(\mathbf{g}) = l_W \circ \mathbf{g} \circ R_{\alpha\beta} : H^+ \to$

$H^+$. Then the Grassmannian tau functions are defined as:

$$\tau_{\alpha\alpha} = \tau = \det T_W(\mathbf{g}),\ \alpha = 1,..,n$$

$$\tau_{\alpha\beta} = \det T_{W_{\alpha\beta}},\ \beta \neq \alpha.$$

The following identity holds for any transformations $\mathbf{g}$ and $\mathbf{g}'$:

$$T_W^{-1}(\mathbf{g})T_W(\mathbf{g}'\mathbf{g}) = T_{W\mathbf{g}^{-1}}(\mathbf{g}')$$

We need that identity only for the special $\mathbf{g}$ and $\mathbf{g}'_\beta$ we have defined. Let us first consider the product $\mathbf{g}'_\alpha\mathbf{g} = G_\alpha(\zeta)\mathbf{g}(t)$. Now we take determinant of both sides of (1.2.1). The left hand side becomes $G_\alpha\tau(t)/\tau(t)$. Now we need to compute the determinant of $T_{W\mathbf{g}^{-1}}(\mathbf{g}'_\alpha)$. Let us find its action on the basis vectors $\epsilon_{\gamma l}$:

$$T_{W\mathbf{g}^{-1}}(\mathbf{g}'_\alpha)\ \epsilon_{\gamma l} = \begin{cases} \epsilon_{\gamma l} & \text{if } \gamma \neq \alpha,\ l = 1,2,.. \\ \epsilon_{\alpha l} - \zeta\epsilon_{\alpha l-1} & \text{if } \gamma = \alpha,\ l \geq 1 \\ \epsilon_{\alpha 1} - \zeta(\mathbf{e}_\alpha - \hat{w}_\alpha) & \text{if } \gamma = \alpha,\ l = 1. \end{cases}$$

where $\hat{w} = w\mathbf{g}^{-1}$ and $\hat{w}_\alpha$ is the $\alpha$- th row of the matrix $\hat{w}$. The matrix element $\hat{w}_{\alpha\beta}$ can be expanded in series in $z$:

$$\hat{w}_{\alpha\beta} = \delta_{\alpha\beta} + \hat{w}^1_{\alpha\beta}z + \hat{w}^2_{\alpha\beta}z^2 + ... + .$$

Now we can compute its determinant in the subspace of $H^+$ where the mapping is different from the identity map:

$$\begin{vmatrix} 1+\hat{w}^1_{\alpha\alpha}\zeta & \hat{w}^2_{\alpha\alpha}\zeta & \hat{w}^3_{\alpha\alpha}\zeta & ... \\ -\zeta & 1 & 0 & ... \\ 0 & -\zeta & 1 & ... \\ ... & ... & ... & ... \end{vmatrix}$$

This determinant can be computed by multiplying the second column by $\zeta$ and adding it to the first one; multiplying the third column by $\zeta$ and adding it to the first and so on. The the determinant of the left hand side of (1.2.1) is

$$1 + \hat{w}^1_{\alpha\alpha}\zeta + \hat{w}^2_{\alpha\alpha}\zeta^2 + ... = \hat{w}_{\alpha\alpha}$$

and we can write an expression for the diagonal Baker function in terms of the tau function:

$$\hat{w}_{\alpha\alpha} = \frac{G_\alpha(\zeta)\tau(t)}{\tau(t)} = \frac{\tau(..,t_{\gamma l} - \delta_{\gamma\alpha} l^{-1}\zeta^l, ...)}{\tau(t)}.$$

We can turn our attention to the non-diagonal case. First we multiply both sides of (1.2.1) by $R_{\alpha\beta}$ to the right and take determinant of both sides. The left hand side will become $G_\beta(\zeta)\tau_{\alpha\beta}(t)/\tau(t)$. The mapping of the the right hand side acts on the basis vectors:

$$T_{W_{\mathbf{g}^{-1}}}(\mathbf{g}'_\beta) \circ R_{\alpha\beta}\ \epsilon_{\gamma l} = \begin{cases} \epsilon_{\gamma l} & \text{if } \gamma \neq \beta,\ l = 1, 2, .. \\ \epsilon_{\alpha l} - \zeta\epsilon_{\beta l-1} & \text{if } \gamma = \beta,\ l \geq 1 \\ -(\mathbf{e}_\alpha - \hat{w}_\alpha) & \text{if } \gamma = \beta,\ l = 1. \end{cases}$$

Its determinant in the subspace of $H^+$ is

$$\begin{vmatrix} w^1_{\alpha\beta} & w^2_{\alpha\beta} & w^3_{\alpha\beta} & \cdots \\ -\zeta & 1 & 0 & \cdots \\ 0 & -\zeta & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

This determinant can be computed the same way as above and it is equal to

$$\hat{w}^1_{\alpha\beta} + \hat{w}^2_{\alpha\beta}\zeta^1 + \hat{w}^3_{\alpha\beta}\zeta^2 ... = \zeta^{-1}\hat{w}_{\alpha\beta}$$

We can express the non-diagonal elements of the Baker function in terms of the tau functions:

$$\hat{w}_{\alpha\beta} = \zeta\frac{G_\beta(\zeta)\tau_{\alpha\beta}(t)}{\tau(t)} = \zeta\frac{\tau_{\alpha\beta}(..,t_{\gamma l} - \delta_{\gamma\beta} l^{-1}\zeta^l, ...)}{\tau(t)}.$$

## 1.3. Some preparational facts for the multi-pole case

We want to define the tau-function as a determinant of some mapping in the subspace $H^{+} = \{\mathbf{f}_k : \mathbf{f}_k \in H_k^{+} \text{ and } \sum_k \mathbf{f}_k(a_k) = 0\}$, and for that reason we shall need a basis in $H^{+}$. Elements of this subspace are sets of $m$ vector rows of length $n$. Let us fix a number $k$, $k = 1, .., m$. We can choose the following basis:

$$\epsilon_{q\gamma l} = \begin{cases} \{\mathbf{0}, ..., (z-a_q)^l \mathbf{e}_\gamma, ..., \mathbf{0}\}, & \text{if } l \geq 1;\ q = 1, ..., m\ ;\ \gamma = 1, ..., n \\ \{\mathbf{0}, ..., \mathbf{e}_\gamma, ... \ \ , -\mathbf{e}_\gamma, ...\mathbf{0}\} & \text{if } l = 0;\ q = 1, ..., m\ , q \neq k;\ \gamma = 1, ..., n; \end{cases}$$

where for $l \geq 1$ only the q-th vector of $\epsilon_{q\gamma l}$ is not zero, and for $l = 0$ the two nonzero entries of $\epsilon_{q\gamma 0}$ are correspondingly at the $q$-th and the $k$-th positions. The vector

$$\mathbf{e}_\gamma = (0, ..., 1, ..., 0)$$

is of length $n$ and its only non-zero entry is at the $\gamma$-th position. Clearly the vectors $\epsilon_{q\gamma l}$ form a basis in $H^{+}$.

**Remark.** The defined basis depends on the number of the pole $k$. Thus for each pole $a_k$ we use its corresponding basis.

We shall also need transformations in $H$ of the form:

$$\mathbf{g} = \{g_1, ..., g_m\},$$

where $g_i$, $i = 1, ..., m$ is a matrix $n \times n$ whose enrties are functions of $z - a_i$. Such a $\mathbf{g}$ acts in $H$ componentwise:

$$\mathbf{g} : \{\mathbf{f}_i\} \mapsto \{\mathbf{f}_i g_i\}$$

(The multiplication on the right-hand side is understood as a vector-row of length $n$ multiplied by a matrix $n \times n$.)

Let $W$ be an element of the Grassmannian and $l_W : H \to H^{+}$ be the projection on $H^{+}$ parallel to $W$. Now consider the mapping

$$T_W(\mathbf{g}) = l_W \circ \mathbf{g} : H^{+} \to H^{+}.$$

If $\mathbf{g}$ and $\mathbf{g}'$ are two transformations as above we can claim :

**Proposition.** $T_W(\mathbf{g}'\mathbf{g}) = T_W(\mathbf{g})T_{W\mathbf{g}^{-1}}(\mathbf{g}')$.

*Proof.* The proof is exactly the same as in [8].

Although the above formula holds for generic transformations $\mathbf{g}$ (with the mild assumption that $\mathbf{g}^{-1}$ exists and $W\mathbf{g}^{-1}$ is an element of the Grassmannian), for our purposes we need them only in a very special form. Namely , let

$$\mathbf{g} = \{e^{\xi_1}, ..., e^{\xi_q}, ..., e^{\xi_m}\},$$

where

$$\xi_q = \sum_{\gamma=1}^{n}\sum_{l=1}^{\infty} t_{q\gamma l}(z-a_q)^{-l}\mathbf{E}_\gamma,$$

$\mathbf{E}_\gamma$ is the $n \times n$ matrix, whose only non-zero entry $E_{\gamma\gamma}$ is equal to 1. Let us also fix an $\alpha$, $\alpha = 1, .., n$ and set

$$\mathbf{g'}_{k\alpha} = \{\mathbf{I}, ..., \mathbf{d}_{k\alpha}, .., \mathbf{I}\},$$

where the diagonal matrix $\mathbf{d}_{k\alpha} = \mathbf{I} - (\zeta - a_k)/(z - a_k)\mathbf{E}_\alpha$ is at the $k$-th position.

**Remark.** Further we will need $\mathbf{d}_{k\alpha}$ written as an exponent:

$$\begin{aligned}\mathbf{d}_{k\alpha} =&\mathbf{I} - (\zeta - a_k)/(z-a_k)\mathbf{E}_\alpha = \exp\{\ln(\mathbf{I} - (\zeta - a_k)/(z-a_k)\mathbf{E}_\alpha)\} = \\ =&\exp\left\{-\sum_{l=1}^{\infty} l^{-1}(\zeta - a_k)^l(z-a_k)^{-l}\mathbf{E}_\alpha\right\}.\end{aligned} \tag{1.3.1}$$

We will need the action of $\mathbf{g'}_{k\alpha}$ on the basis $\epsilon_{q\gamma l}$:

$$\mathbf{g'}_{k\alpha}\epsilon_{q\gamma l} = \begin{cases} \epsilon_{q\gamma l} & \text{if } \{\gamma \neq \alpha,\ q = 1, .., m,\ l = 0, 1, ...\} \text{ or} \\ & \text{if } \{q \neq k,\ \gamma = 1, .., n,\ l = 1, 2, ...\} \\ \epsilon_{k\alpha l} - (\zeta - a_k)\epsilon_{k\alpha l-1} & \text{if } \{l > 1,\ q = k,\ \gamma = \alpha\} \\ \epsilon_{k\alpha 1} - (\zeta - a_k)\tilde{\epsilon}_{k\alpha 0} & \text{if } \{l = 1,\ q = k,\ \gamma = \alpha\} \\ \epsilon_{k\alpha 0} + (\zeta - a_k)\tilde{\epsilon}_{k\alpha -1} & \text{if } \{l = 0,\ q = 1, .., m;\ q \neq k,\ \gamma = \alpha\}, \end{cases}$$

where

$$\begin{aligned}\tilde{\epsilon}_{k\alpha 0} =&\{\mathbf{0}, ..., \quad \mathbf{e}_\alpha, .. \quad, \mathbf{0}\}, \\ \tilde{\epsilon}_{k\alpha -1} =&\{\mathbf{0}, ..., (z-a_k)^{-1}\mathbf{e}_\alpha, .., \mathbf{0}\}.\end{aligned}$$

The only non-zero entry of either vector is at the $k$-th position. We will also need the vectors:

$$\tilde{\epsilon}_{q\alpha 0} = \{\mathbf{0}, \ldots, \quad \mathbf{e}_\alpha, .. \quad, \mathbf{0}\},$$

$$\tilde{\epsilon}_\alpha = \{\mathbf{e}_\alpha, \ldots, \quad \mathbf{e}_\alpha, .., \quad \mathbf{e}_\alpha\},$$

where the only non-zero entry of $\tilde{\epsilon}_{q\alpha 0}$ is at the $q$-th position and $q \neq k$.

**Remark.** None of the vectors with tilde belongs to $H^+$, however they play an important role in our construction and are convenient for computations.

Let $W$ be an element of the Grassmannian $Gr$ and $w = \{w_k\}$ be the corresponding Baker function. Let also $\hat{w} = w\mathbf{g}$, where $\mathbf{g}$ was defined above. Now consider the set of all $\alpha$-th rows of the Baker function $\hat{w}$:

$$\hat{w}_\alpha = \{\hat{w}_{1,\alpha}, .., \hat{w}_{q,\alpha}, .., \hat{w}_{m,\alpha}\},$$

where $\hat{w}_{q,\alpha}$ is the $\alpha$-th row of the matrix $\hat{w}_q$:

$$\hat{w}_{q,\alpha} = (\hat{w}_{q,\alpha 1}, \ldots, \hat{w}_{q,\alpha\gamma}, .., \hat{w}_{q,\alpha n}).$$

The vector $\hat{w}_{q,\alpha}$ can be expanded in series in $z - a_q$ $(q = 1, .., m)$:

$$\hat{w}_{q,\alpha} = \tilde{\epsilon}_{q,\alpha 0} + \hat{w}^0_{q,\alpha} + \hat{w}^1_{q,\alpha}(z - a_q) + \ldots .$$

Now we can define $\hat{w}^0_\alpha$:

$$\hat{w}^0_\alpha = \{\hat{w}^0_{1,\alpha}, .., \hat{w}^0_{q,\alpha}, .., \hat{w}^0_{m,\alpha}\}.$$

Since $\sum_{q=1}^m \hat{w}^0_{q,\alpha} = \mathbf{0}$ we have that $\hat{w}^0_\alpha$ belongs to $H^+$. We shall need the following lemma:

**Lemma 1.3.1.** The following identity is valid:

$$\hat{w}^0_\alpha = \sum_{q \neq k} \sum_{\gamma=1}^n \hat{w}^0_{q,\alpha\gamma} \epsilon_{q,\alpha 0}.$$

*Proof.* First we can expand vector $\hat{w}^0_{q,\alpha}$

$$\hat{w}^0_{q,\alpha} = (\hat{w}^0_{q,\alpha 1}, .., \hat{w}^0_{q,\alpha\gamma}, .., \hat{w}^0_{q,\alpha n}) = \sum_{\gamma=1}^n (0, \ldots, \hat{w}^0_{q,\alpha\gamma}, \ldots, 0) = \sum_{\gamma=1}^n \hat{w}^0_{q,\alpha\gamma} \mathbf{e}_\gamma.$$

From $\sum_{q=1}^{m} \hat{w}^0_{q,\alpha} = \mathbf{0}$ it follows that $\hat{w}^0_{k,\alpha} = -\sum_{q\neq k} \hat{w}^0_{q,\alpha}$. Now we can write:

$$\hat{w}^0_\alpha = \{\hat{w}^0_{1,\alpha}, .., \hat{w}^0_{q,\alpha}, .., \hat{w}^0_{k,\alpha}, .., \hat{w}^0_{m,\alpha}\} = \{\hat{w}^0_{1,\alpha}, .., \hat{w}^0_{q,\alpha}, .., -\sum_{q\neq k} \hat{w}^0_{q,\alpha}, .., \hat{w}^0_{m,\alpha}\} =$$

$$= \sum_{q\neq k} \{\mathbf{0}, ..., \hat{w}^0_{q,\alpha}, ..., -\hat{w}^0_{q,\alpha}, ..., \mathbf{0}\} =$$

$$= \sum_{q\neq k} \{\mathbf{0}, ..., \sum_{\gamma=1}^{n} \hat{w}^0_{q,\alpha\gamma} \mathbf{e}_0, ..., -\sum_{\gamma=1}^{n} \hat{w}^0_{q,\alpha\gamma} \mathbf{e}_0, ..., \mathbf{0}\} =$$

$$= \sum_{q\neq k} \sum_{\gamma=1}^{n} \hat{w}^0_{q,\alpha\gamma} \{\mathbf{0}, ..., \mathbf{e}_\gamma, ..., -\mathbf{e}_\gamma, ..., \mathbf{0}\} = \sum_{q\neq k} \sum_{\gamma=1}^{n} \hat{w}^0_{q,\alpha\gamma} \epsilon_{q\gamma 0}.$$

The proof of the lemma is completed.

We will need a few more identities.

**Lemma 1.3.2.** The following identies hold:

$$\begin{aligned}
\epsilon_{q\alpha 0} &= \tilde{\epsilon}_{q\alpha 0} - \tilde{\epsilon}_{k\alpha 0} \\
\tilde{\epsilon}_{k\alpha 0} &= \frac{1}{m}(\tilde{\epsilon}_\alpha - \sum_{q\neq k} \epsilon_{q\alpha 0}) \\
\tilde{\epsilon}_{k\alpha 0} &= \frac{1}{m}(\tilde{\epsilon}_\alpha - \hat{w}_\alpha) - \frac{1}{m} \sum_{q\neq k} \epsilon_{q\alpha 0} + \frac{1}{m} \hat{w}_\alpha.
\end{aligned}$$

*Proof.* The first identity is obvious. The second identity follows from the first one:

$$\begin{aligned}
\tilde{\epsilon}_\alpha &= \sum_{q=1}^{m} \tilde{\epsilon}_{q\alpha 0} = \sum_{q\neq k}^{m} \tilde{\epsilon}_{q\alpha 0} \ + \ \tilde{\epsilon}_{k\alpha 0} \\
&= \sum_{q\neq k}^{m} (\tilde{\epsilon}_{q\alpha 0} + \tilde{\epsilon}_{k\alpha 0}) \ + \ \tilde{\epsilon}_{k\alpha 0} = \sum_{q\neq k}^{m} \epsilon_{q\alpha 0} \ + \ m\tilde{\epsilon}_{k\alpha 0}.
\end{aligned}$$

Solving for $\tilde{\epsilon}_{k\alpha 0}$ from the last expression we obtain the identity. The third identity is obtained from the second one by adding and subtracting $\hat{w}_\alpha/m$. The lemma is proved.

The last identity of Lemma 1.3.2 is very useful because we will need to write vectors in $H$ according to the decomposition $H^+ \oplus W\mathbf{g}^{-1}$.

## 1.4. Diagonal tau functions for the multi-pole hierarchy

Consider the mapping $T_W(\mathbf{g}) = l_W \circ \mathbf{g}$ from the previous section. The determinant of this mapping $\tau = \det T_W(\mathbf{g})$ will be the general (or denominator) tau function – just like the special case of a single pole. The similarity with the single pole case ends here. As we saw in section 1.2 in that case all the diagonal tau functions $\tau_{\alpha\alpha}$ coincide and are equal to $\tau$. Here the diagonal tau functions $\tau_{k,\alpha\alpha}$ are different for different indices $k$ or $\alpha$.

**Remark.** The main difference between the cases of a single pole and many poles is in the subspace $H_c^+ = \{\mathbf{c} = \text{const} : \sum_i \mathbf{c}_i = 0\}$ of $H^+$. (In the case of a single pole there are no non-zero constants in $H^+$.)

Consider the mapping $R_{k\alpha\alpha} : H^+ \to H$:

$$R_{k\alpha\alpha} : \begin{cases} \epsilon_{q\alpha 0} \mapsto \tilde{\epsilon}_{q\alpha 0} & q \neq k \\ \epsilon_{q\gamma l} \mapsto \tilde{\epsilon}_{q\gamma l} & \text{for all other indices.} \end{cases}$$

As we see from the definition $R_{k\alpha\alpha}$ is almost an identity map except for the vectors $\epsilon_{q\alpha 0}$:

$$R_{k\alpha\alpha} : \{\mathbf{0}, ..., \mathbf{e}_\alpha, ..., -\mathbf{e}_\alpha, .., \mathbf{0}\} \mapsto \{\mathbf{0}, ..., \mathbf{e}_\alpha, ..., \mathbf{0}, .., \mathbf{0}\}.$$

Now we consider the mapping

$$T_W(\mathbf{g}) \circ R_{k\alpha\alpha} = l_W \circ \mathbf{g} \circ R_{k\alpha\alpha} \ : H^+ \longrightarrow H^+.$$

The determinant of this mapping will be the diagonal tau-function $\tau_{k\alpha\alpha}$ of the hierarchy.

**Theorem 1.4.1** Let $W$ be an element of the Grassmannian $Gr$ and the corresponding $w_W$ be a Baker function for the multipole ($m > 1$) ZS hierarchy. The maps $\mathbf{g}$, $\mathbf{g}'_{k\alpha}$, $T_W(\mathbf{g})$, $R_{k\alpha\alpha}$ are as above. Let

$$\tau(t) = (\det T_W(\mathbf{g}))\,(t), \ \ \tau_{k\alpha\alpha} = (\det T_W(\mathbf{g}) \circ R_{k\alpha\alpha})\,(t).$$

Then the diagonal elements $\hat{w}_{k\alpha\alpha}$ of the Baker function can be expressed in terms of the functions $\tau$ and $\tau_{k\alpha\alpha}$:

$$\hat{w}_{k,\alpha\alpha}(t,z) = m\frac{\tau_{k,\alpha\alpha}(..,t_{q\gamma l} - \delta_{kq}\delta_{\alpha\gamma}l^{-1}(z-a_k)^l,..)}{\tau(t)}.$$

*Proof.* As in AKNS central to our proof is the formula:

$$T_W(\mathbf{g}'\mathbf{g}) \;=\; T_W(\mathbf{g})T_{W\mathbf{g}^{-1}}(\mathbf{g}'). \tag{1.4.1}$$

We multiply both sides of the formula first to the left by $T_W(\mathbf{g}^{-1})$, and to the right by $R_{k\alpha\alpha}$:

$$T_W(\mathbf{g}^{-1})T_W(\mathbf{g}'\mathbf{g})R_{k\alpha\alpha} = T_{W\mathbf{g}^{-1}}(\mathbf{g}')R_{k\alpha\alpha}. \tag{1.4.2}$$

Let us first consider the left hand side of (1.4.1). Using equality (1.3.1) we compute $\mathbf{g}\mathbf{g}'_{k\alpha}$:

$$\begin{aligned}
\mathbf{g}\mathbf{g}'_{k\alpha} &= \{\exp\xi_1,...,\exp\xi_k,...,\exp\xi_m\}\{\mathbf{I},...,\mathbf{I},...,\mathbf{I}-(\zeta-a_k)/(z-a_k)\mathbf{E}_\alpha,...,\mathbf{I}\} = \\
&= \{\exp\xi_1,...,\exp\xi_k(\mathbf{I}-(\zeta-a_k)/(z-a_k)\mathbf{E}_\alpha),...,\exp\xi_m\} = \\
&= \{\exp\xi_1,...,\exp\sum_{\gamma=1}^{n}\sum_{l=1}^{\infty}\Big(t_{k\gamma l}-\delta_{\gamma\alpha}l^{-1}(\zeta-a_k)^l\Big)(z-a_k)^{-l}\mathbf{E}_\gamma,...,\exp\xi_m\} = \\
&= \mathbf{g}(...,t_{q\gamma l}-\delta_{qk}\delta_{\gamma\alpha}l^{-1}(\zeta-a_q)^l,...) = G_{k\alpha}(\zeta)\mathbf{g}(t).
\end{aligned}$$

If we take determinant of both sides of equation (1.4.2) the resulting left hand side will be just what we want: $G_{k\alpha}(\zeta)\tau_{k\alpha\alpha}(t)/\tau(t)$. Now we need to compute the determinant of the rihgt hand side of (1.4.2). Let us for convenience denote that mapping by $A_{k\alpha\alpha}$:

$$A_{k\alpha\alpha} = l_{W\mathbf{g}^{-1}} \circ \mathbf{g}'_{k\alpha} \circ R_{k\alpha\alpha}.$$

Let us compute the action $A_{k\alpha\alpha}$ on the basis in $H^+$:

$$\epsilon_{q\alpha 0} \overset{R_{k\alpha\alpha}}{\longmapsto} \tilde{\epsilon}_{q\alpha 0} \overset{\mathbf{g}'_{k\alpha}}{\longmapsto} \tilde{\epsilon}_{q\alpha 0} \overset{l_{W\mathbf{g}^{-1}}}{\longmapsto} \epsilon_{q\alpha 0} + \frac{1}{m}(\tilde{\epsilon}_\alpha - \hat{w}_\alpha) - \frac{1}{m}\sum_{q'\neq k}\epsilon_{q'\alpha 0}, \qquad \text{for } q\neq k;$$

$$\epsilon_{k\alpha 1} \overset{R_{k\alpha\alpha}}{\longmapsto} \epsilon_{k\alpha 1} \overset{\mathbf{g}'_{k\alpha}}{\longmapsto} \epsilon_{k\alpha 1} - \zeta_k\tilde{\epsilon}_{k\alpha 0} \overset{l_{W\mathbf{g}^{-1}}}{\longmapsto} \epsilon_{k\alpha 1} - \frac{\zeta_k}{m}\Big((\tilde{\epsilon}_\alpha - \hat{w}_\alpha) - \sum_{q'\neq k}\epsilon_{q'\alpha 0}\Big);$$

$$\epsilon_{k\alpha l} \overset{R_{k\alpha\alpha}}{\longmapsto} \epsilon_{k\alpha l} \overset{\mathbf{g}'_{k\alpha}}{\longmapsto} \epsilon_{k\alpha l} - \zeta_k\epsilon_{k\alpha\; l-1} \overset{l_{W\mathbf{g}^{-1}}}{\longmapsto} \epsilon_{k\alpha l} - \zeta_k\epsilon_{k\alpha\; l-1}, \qquad \text{for } l\geq 1;$$

$$\epsilon_{q\gamma l} \overset{R_{k\alpha\alpha}}{\longmapsto} \epsilon_{q\gamma l} \overset{\mathbf{g}'_{k\alpha}}{\longmapsto} \epsilon_{q\gamma l} \overset{l_{W\mathbf{g}^{-1}}}{\longmapsto} \epsilon_{q\gamma l}, \qquad \text{for all other basis vectors,}$$

where $\zeta_k = \zeta - a_k$. In computing the first two vectors we used Lemma 1.3.2. We now expand the vectors $A_{k\alpha\alpha}\epsilon_{k\alpha 1}$ and $A_{k\alpha\alpha}\epsilon_{q\alpha 0}$ in the basis of the subspace $H^+/H_A$ ($H_A = \{\mathbf{x} \in H^+ : \ A_{k\alpha\alpha}\mathbf{x} = \mathbf{x}\}$ is the subspace where $A_{k\alpha\alpha}$ acts as an identity map).

$$A_{k\alpha\alpha}\epsilon_{k\alpha 1} = \epsilon_{k\alpha 1} - \frac{\zeta_k}{m}\Big((\tilde{\epsilon}_\alpha - \hat{w}_\alpha) - \sum_{q'\neq k} \epsilon_{q'\alpha 0}\Big) =$$

$$= \epsilon_{k\alpha 1} + \frac{\zeta_k}{m}\Big(\sum_{q'\neq q} \hat{w}^0_{q'\alpha\alpha}\epsilon_{q'\alpha 0} + \sum_{l=1}^{\infty} \hat{w}^l_{k\alpha\alpha}\epsilon_{k\alpha l} + \sum_{q'\neq k} \epsilon_{q'\alpha 0}\Big) =$$

$$= \frac{\zeta_k}{m}\sum_{q'\neq q}(1 + \hat{w}^0_{q'\alpha\alpha})\epsilon_{q'\alpha 0} + \left(1 + \frac{\zeta_k}{m}\hat{w}^1_{k\alpha\alpha}\right)\epsilon_{k\alpha 1} + \frac{\zeta_k}{m}\sum_{l=2}^{\infty}\hat{w}^l_{k\alpha\alpha}\epsilon_{k\alpha l}.$$

In expanding $\tilde{\epsilon}_\alpha - \hat{w}_\alpha$ we used Lemma 1.3.1 from the previous section. Similarly we obtain:

$$A_{k\alpha\alpha}\epsilon_{q\alpha 0} = \epsilon_{q\alpha 0} - \frac{1}{m}\sum_{q'\neq q}(1 + \hat{w}^0_{q'\alpha\alpha})\epsilon_{q'\alpha 0} - \frac{1}{m}\sum_{l=1}^{\infty}\hat{w}^l_{k\alpha\alpha}\epsilon_{k\alpha l}.$$

Then the determinant of $A_{k\alpha\alpha}$ looks like:

$$\begin{vmatrix} 1 - \frac{1}{m}(1+\hat{w}^0_{1\alpha\alpha}) & \dots & -\frac{1}{m}(1+\hat{w}^0_{m\alpha\alpha}) & -\frac{1}{m}\hat{w}^1_{k\alpha\alpha} & -\frac{1}{m}\hat{w}^2_{k\alpha\alpha} & \dots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \\ -\frac{1}{m}(1+\hat{w}^0_{1\alpha\alpha}) & \dots & 1 - \frac{1}{m}(1+\hat{w}^0_{m\alpha\alpha}) & -\frac{1}{m}\hat{w}^1_{k\alpha\alpha} & -\frac{1}{m}\hat{w}^2_{k\alpha\alpha} & \dots \\ \frac{\zeta_k}{m}(1+\hat{w}^0_{1\alpha\alpha}) & \dots & \frac{\zeta_k}{m}(1+\hat{w}^0_{m\alpha\alpha}) & 1 + \frac{\zeta_k}{m}\hat{w}^1_{k\alpha\alpha} & \frac{\zeta_k}{m}\hat{w}^2_{k\alpha\alpha} & \dots \\ 0 & \dots & 0 & -\zeta_k & 1 & \dots \\ 0 & \dots & 0 & 0 & -\zeta_k & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

To each of the first $m-1$ rows (vectors $A_{k\alpha\alpha}\epsilon_{q\alpha 0}$) of the determinant add the $m$-th row (vector $A_{k\alpha\alpha}\epsilon_{k\alpha 1}$) divided by $\zeta_k$. We obtain

$$\left|\begin{array}{cccccc} 1 & \dots & 0 & \zeta_k^{-1} & 0 & \dots \\ \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & \zeta_k^{-1} & 0 & \dots \\ \frac{\zeta_k}{m}(1+\hat{w}^0_{1\alpha\alpha}) & \dots & \frac{\zeta_k}{m}(1+\hat{w}^0_{m\alpha\alpha}) & 1+\frac{\zeta_k}{m}\hat{w}^1_{k\alpha\alpha} & \frac{\zeta_k}{m}\hat{w}^2_{k\alpha\alpha} & \dots \\ 0 & \dots & 0 & -\zeta_k & 1 & \dots \\ 0 & \dots & 0 & 0 & -\zeta_k & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{array}\right| .$$

Now we multiply each of the first $m-1$ columns by $-\zeta_k^{-1}$ and then add all of them to the $m$-th column. All the entries $(j, m-1)$, $(j = 1, .., m-1)$ will become zero, and the diagonal entry $(m, m)$ will look like:

$$\begin{aligned} (A_{k\alpha\alpha})_{mm} &= 1 + \frac{\zeta_k}{m}\hat{w}^1_{k\alpha\alpha} - \frac{1}{\zeta_k}\frac{\zeta_k}{m}\sum_{q'\neq q}(1+\hat{w}^0_{q'\alpha\alpha}) = \\ &= 1 + \frac{\zeta_k}{m}\hat{w}^1_{k\alpha\alpha} - \frac{m-1}{m} - \frac{1}{m}\sum_{q'\neq q}\hat{w}^0_{q'\alpha\alpha} = \\ &= \frac{1}{m}(1 + \hat{w}^0_{k\alpha\alpha} + \hat{w}^1_{k\alpha\alpha}\zeta_k) \end{aligned}$$

Thus we have

$$\left|\begin{array}{cccccc} 1 & \dots & 0 & 0 & 0 & \dots \\ \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots \\ \frac{\zeta_k}{m}(1+\hat{w}^0_{1\alpha\alpha}) & \dots & \frac{\zeta_k}{m}(1+\hat{w}^0_{m\alpha\alpha}) & \frac{1}{m}(1+\hat{w}^0_{k\alpha\alpha}+\hat{w}^1_{k\alpha\alpha}\zeta_k) & \frac{\zeta_k}{m}\hat{w}^2_{k\alpha\alpha} & \dots \\ 0 & \dots & 0 & -\zeta_k & 1 & \dots \\ 0 & \dots & 0 & 0 & -\zeta_k & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{array}\right| .$$

Using the Lagrange theorem we reduce the determinant to

$$\begin{vmatrix} \frac{1}{m}(1+\hat{w}^0_{k\alpha\alpha}+\hat{w}^1_{k\alpha\alpha}\zeta_k) & \frac{1}{m}\hat{w}^2_{k\alpha\alpha}\zeta_k & \frac{1}{m}\hat{w}^3_{k\alpha\alpha}\zeta_3 & \ldots \\ -\zeta_k & 1 & 0 & \ldots \\ 0 & -\zeta_k & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

The last determinant looks like the one in the single pole case and it is equal to

$$\frac{1}{m}(1+\hat{w}^0_{k\alpha\alpha}+\hat{w}^1_{k\alpha\alpha}\zeta_k+\hat{w}^2_{k\alpha\alpha}\zeta_k^2+\hat{w}^3_{k\alpha\alpha}\zeta_k^3+\ldots)=\frac{1}{m}\hat{w}_{k\alpha\alpha}(t,\zeta_k),$$

which completes the proof of the theorem.

**Remark.** We could have chosen the transformations $\mathbf{g}$ and $\mathbf{g}'_{k\alpha}$ differently:

$$\begin{aligned} \mathbf{g} &= \{\exp\xi,\ldots,\exp\xi,\ldots,\exp\xi\}, \text{ where } \xi=\sum_{i=1}^{m}\xi_i; \\ \mathbf{g}'_{k\alpha} &= \{\mathbf{d}_{k\alpha},\ldots,\mathbf{d}_{k\alpha},\ldots,\mathbf{d}_{k\alpha}\}. \end{aligned}$$

Then instead of an expression for $\hat{w}_{k\alpha\alpha}$ we would have a formula for $\hat{\hat{w}}_{k\alpha\alpha}$. (Recall that $\hat{\hat{w}}_k = w_k \exp\xi$.) The corresponding relation will look like:

$$\hat{\hat{w}}_{k,\alpha\alpha}(t,z)=m\frac{\tau_{k,\alpha\alpha}(..,t_{q\gamma l}-\delta_{kq}\delta_{\alpha\gamma}l^{-1}(z-a_k)^l,..)}{\tau(t)}\prod_{q\neq k}\frac{a_k-a_q}{z-a_q}.$$

## 1.5. Discussion of the non-diagonal case.

If we follow the logic of the single pole non-diagonal case we can consider a mapping $R_{k\alpha\beta} : H^+ \to H$ $(\beta \neq \alpha)$ :

$$R_{k\alpha\beta} : \begin{cases} \epsilon_{q\beta 0} \mapsto \bar{\epsilon}_{q\beta 0} & q \neq k \\ \epsilon_{k\beta 1} \mapsto -\bar{\epsilon}_{\alpha} & \\ \epsilon_{q\gamma l} \mapsto \bar{\epsilon}_{q\gamma l} & \text{for all other indices.} \end{cases}$$

We then consider the function

$$\tau_{k\alpha\beta}(t) = \det\left(T_W(\mathrm{g}) \circ R_{k\alpha\beta}\right)(t)$$

as a candidate for tau-function. Once again using the identity (1.4.1), multiplying its both sides by $R_{k\alpha\beta}$ to the right, and taking a determinant of both sides we obtain on the left hand side $G_{k\beta}(\zeta)\tau_{k\alpha\beta}(t)/\tau(t)$. However, the determinant of the right hand side will be (after some computations similar to those in section 1.4)

$$\frac{1}{m\zeta}(\hat{w}_{k\alpha\beta}(\zeta)(1+\hat{w}^0_{k\beta\beta}) - \hat{w}_{k\beta\beta}(\zeta)\hat{w}^0_{k\alpha\beta}),$$

which is not what we want. We got a mixture of diagonal and non-diagonal Baker functions, which can not be separated. Perhaps, one needs to extend the space $H^+$ and define corresponding mappings there.

## 1.6. Appendix: A useful expansion.

In the previous sections we saw the importance of representing vectors from $H$ in the form $H^+ \oplus W\mathbf{g}^{-1}$ - we can compute the action of the projection $l_{W\mathbf{g}^{-1}}$ : $H \to H^+$. In Lemma 1.3.2 we saw how to do that for the vectors $\bar{\epsilon}_{k\alpha 0}$ and $\bar{\epsilon}_{q\alpha 0}$. Now we will see how represent vectors with negative powers in $z - a_k$. We start with the lowest negative power - vector $\bar{\epsilon}_{k\alpha-1}$. Our expansion will be modulo the subspace spanned by the following vectors: $\epsilon_{q\gamma 0}$ for $\gamma \neq \alpha$, $q \neq k$; $\epsilon_{q\gamma l}$ for $\gamma \neq \alpha$, $l \geq 1$ or $q \neq k$, $l \geq 1$; and $W\mathbf{g}^{-1}$. Let us for convenience denote $z_k = z - a_k$.

$$
\begin{aligned}
\bar{\epsilon}_{k\alpha-1} &= z_k^{-1}\bar{\epsilon}_{k\alpha 0} = z_k^{-1}\left(\frac{1}{m}(\bar{\epsilon}_\alpha - \hat{w}_\alpha) - \frac{1}{m}\sum_{q\neq k}\epsilon_{q\alpha 0} + \frac{1}{m}\hat{w}_\alpha\right) = \\
&= -\frac{1}{m}z_k^{-1}\sum_{q\neq k}(1+\hat{w}^0_{q\alpha\alpha})\epsilon_{q\alpha 0} - \frac{1}{m}z_k^{-1}\sum_{l=1}^{\infty}\hat{w}^l_{k\alpha\alpha}\epsilon_{k\alpha l} = \\
&= -\frac{1}{m}z_k^{-1}\sum_{q\neq k}(1+\hat{w}^0_{q\alpha\alpha})(\bar{\epsilon}_{q\alpha 0} - \bar{\epsilon}_{k\alpha 0}) - \frac{1}{m}\bar{\epsilon}_{k\alpha 0} - \frac{1}{m}\sum_{l=1}^{\infty}\hat{w}^{l+1}_{k\alpha\alpha}\epsilon_{k\alpha l} = \\
&= \overbrace{-\frac{1}{m}z_k^{-1}\sum_{q\neq k}(1+\hat{w}^0_{q\alpha\alpha})\bar{\epsilon}_{q\alpha 0}}^{I} + \overbrace{\frac{1}{m}z_k^{-1}\sum_{q\neq k}(1+\hat{w}^0_{q\alpha\alpha})\bar{\epsilon}_{k\alpha 0}}^{II} + \\
&\overbrace{-\frac{1}{m}\bar{\epsilon}_{k\alpha 0} - \frac{1}{m}\sum_{l=1}^{\infty}\hat{w}^{l+1}_{k\alpha\alpha}\epsilon_{k\alpha l}}^{III}.
\end{aligned}
\tag{1.6.1}
$$

Now consider the above expressions one by one:

$$
\begin{aligned}
I &= -\frac{1}{m} z_k^{-1} \sum_{q\neq k} (1+\hat{w}^0_{q\alpha\alpha})\bar{\epsilon}_{q\alpha 0} = \\
&= -\frac{1}{m} \sum_{q\neq k} (1+\hat{w}^0_{q\alpha\alpha}) \left( \frac{1}{a_q - a_k} \bar{\epsilon}_{q\alpha 0} + \sum_{l=1}^{\infty} \frac{1}{(a_q - a_k)^{l+1}} \epsilon_{q\alpha l} \right) = \\
&= -\frac{1}{m} \sum_{q\neq k} \frac{1+\hat{w}^0_{q\alpha\alpha}}{a_q - a_k} \bar{\epsilon}_{q\alpha 0} = -\frac{1}{m} \sum_{q\neq k} \frac{1+\hat{w}^0_{q\alpha\alpha}}{a_q - a_k} (\epsilon_{k\alpha 0} - \bar{\epsilon}_{k\alpha 0}) = \\
&= -\frac{1}{m} \sum_{q\neq k} \frac{1+\hat{w}^0_{q\alpha\alpha}}{a_q - a_k} \epsilon_{q\alpha 0} \\
&- \frac{1}{m} \left( \sum_{q\neq k} \frac{1+\hat{w}^0_{q\alpha\alpha}}{a_q - a_k} \right) \left( (\frac{1}{m}\bar{\epsilon}_\alpha - \hat{w}_\alpha) - \frac{1}{m} \sum_{q\neq k} \epsilon_{q\alpha 0} + \frac{1}{m} \hat{w}_\alpha \right) = \\
&= -\frac{1}{m} \sum_{q\neq k} \frac{1+\hat{w}^0_{q\alpha\alpha}}{a_q - a_k} \epsilon_{q\alpha 0} - \frac{1}{m^2} \left( \sum_{q\neq k} \frac{1+\hat{w}^0_{q\alpha\alpha}}{a_q - a_k} \right) \sum_{q\neq k} (1+\hat{w}^0_{q\alpha\alpha}) \epsilon_{q\alpha 0} + \\
&+ \frac{1}{m^2} \left( \sum_{q\neq k} \frac{1+\hat{w}^0_{q\alpha\alpha}}{a_q - a_k} \right) \sum_{l=1}^{\infty} \hat{w}^l_{q\alpha\alpha} \epsilon_{k\alpha l}.
\end{aligned}
\tag{1.6.2}
$$

Now we consider the second expression from (1.6.1):

$$
\begin{aligned}
II &= \frac{1}{m} z_k^{-1} \sum_{q\neq k} (1+\hat{w}^0_{q\alpha\alpha})\bar{\epsilon}_{k\alpha 0} = \frac{1}{m} \left( \sum_{q\neq k} 1 + \frac{1}{m} \sum_{q\neq k} \hat{w}^0_{q\alpha\alpha} \right) \bar{\epsilon}_{k\alpha -1} = \\
&= \frac{1}{m} (m - 1 - \hat{w}^0_{k\alpha\alpha}) \bar{\epsilon}_{k\alpha -1}.
\end{aligned}
\tag{1.6.3}
$$

If we transfer $II$ to the left hand side of (1.6.1) we obtain

$$
\left( 1 - \frac{m-1}{m} + \frac{1}{m} \hat{w}^0_{k\alpha\alpha} \right) \bar{\epsilon}_{k\alpha -1} = \frac{1+\hat{w}^0_{k\alpha\alpha}}{m} \bar{\epsilon}_{k\alpha -1}. \tag{1.6.4}
$$

Let us rewrite $III$:

$$
\begin{aligned}
III &= -\frac{1}{m} \hat{w}^1_{k\alpha\alpha} \bar{\epsilon}_{k\alpha -0} = -\frac{1}{m^2} \hat{w}^1_{k\alpha\alpha} \Big( (\bar{\epsilon}_\alpha - \hat{w}_\alpha) - \sum_{q\neq k} \epsilon_{q\alpha 0} + \hat{w}_\alpha \Big) = \\
&= \frac{\hat{w}^1_{k\alpha\alpha}}{m^2} \sum_{q\neq k} (1+\hat{w}^0_{q\alpha\alpha}) \epsilon_{q\alpha 0} + \frac{\hat{w}^1_{k\alpha\alpha}}{m^2} \sum_{l=1}^{\infty} \hat{w}^l_{k\alpha\alpha} \epsilon_{k\alpha l}.
\end{aligned}
\tag{1.6.5}
$$

We use (1.6.2) and (1.6.5) to obtain

$$
\begin{aligned}
I + III = \frac{1}{m^2} \sum_{q \neq k} (1 + \hat{w}^0_{q\alpha\alpha}) \left( \hat{w}^1_{k\alpha\alpha} + \sum_{q \neq k} \frac{1 + \hat{w}^0_{q\alpha\alpha}}{a_q - a_k} - \frac{m}{a_q - a_k} \right) \epsilon_{q\alpha 0} \\
+ \frac{1}{m^2} \sum_{l=1}^{\infty} \hat{w}^l_{k\alpha\alpha} \left( \hat{w}^1_{k\alpha\alpha} + \sum_{q \neq k} \frac{1 + \hat{w}^0_{q\alpha\alpha}}{a_q - a_k} \right) \epsilon_{k\alpha l}.
\end{aligned}
\tag{1.6.6}
$$

Finally from (1.6.4) and (1.6.6) we have

$$
\begin{aligned}
\bar{\epsilon}_{k\alpha-1} = \frac{1}{m} \sum_{q \neq k} \frac{1 + \hat{w}^0_{q\alpha\alpha}}{1 + \hat{w}^0_{k\alpha\alpha}} \left( \hat{w}^1_{k\alpha\alpha} + \sum_{q \neq k} \frac{1 + \hat{w}^0_{q\alpha\alpha}}{a_q - a_k} - \frac{m}{a_q - a_k} \right) \epsilon_{q\alpha 0} \\
+ \frac{1}{m} \sum_{l=1}^{\infty} \frac{\hat{w}^l_{k\alpha\alpha}}{1 + \hat{w}^0_{k\alpha\alpha}} \left( \hat{w}^1_{k\alpha\alpha} + \sum_{q \neq k} \frac{1 + \hat{w}^0_{q\alpha\alpha}}{a_q - a_k} \right) \epsilon_{k\alpha l}.
\end{aligned}
\tag{1.6.7}
$$

Now we are able to express any negative power in a similar fashion, e.g. $\bar{\epsilon}_{k\alpha-2} = z_k^{-1} \bar{\epsilon}_{k\alpha-1}$. Then use (1.6.7) and repeat the procedure all over again.

# Chapter II

# ALGEBRAIC-GEOMETRICAL TAU FUNCTIONS FOR THE HIERARCHY GZS

## 2.1. Some facts and definitions

In this section we shall recall necessary facts and definitions from the theory of Riemann surfaces. We adopt the notation from [10]. More detail and proofs can be found in [10], [14] etc.

Let $\Gamma$ be a (compact) Riemann surface of genus $g > 0$. We say that $\{a_i,\ b_i;\ i = 1, ..., g\}$ is a canonical system of closed contours on the Riemann surface if:

i) any contour $\gamma$ on $\Gamma$ can be written as an integral linear combination

$$\gamma \sim \sum_{i=1}^{g} m_i a_i + n_i b_i, \text{ and}$$

ii) their intersection numbers are

$$a_i \circ a_j = b_i \circ b_j = 0,\ a_i \circ b_j = \delta_{ij},\ i, j = 1, .., g.$$

An abelian differential of first type is a differential on $\Gamma$ that can be written in a form $\omega = \phi(z)dz$ in a neighborhood of any point, where $z$ is a local parameter and $\phi(z)$ is a holomorphic function. It can be proved that holomorphic differentials exist on any Riemann surface ($g > 0$) and they form a linear space of dimension $g$. A basis of holomorphic differentials $\{\omega_i,\ i = 1, 2, ..., g\}$ is called normalized (with respect to the canonical basis $\{a_i,\ b_i;\ i = 1, ..., g\}$) if their $a$-periods are

$$\int_{a_i} \omega_j = 2\pi i \delta_{ij}.$$

Let $B = (B_{ij}) = (\int_{b_i} \omega_j)$ be the matrix of their $b$-periods. ($B$ is referred to as the matrix of the periods of the Riemann surface.) The matrix $B$ is symmetric and its real part $Re(B)$ is negative definite.

Let $\mathbf{e}_j,\ j = 1,..,g$ be the vector in $\mathbf{C}^g$ whose only non-zero entry is 1 at the $j$-th position. It can be shown that the vectors $\{2\pi i\mathbf{e}_j,\ B\mathbf{e}_j,\ j = 1,..,g\}$ are linearly independent (over $\mathbf{R}$). Let $\Lambda$ be the integral lattice spanned by these vectors. Then the torus $J(\Gamma) = \mathbf{C}^g/\Lambda$ is called the Jacobian of the Riemann surface $\Gamma$. The Abel map $\mathcal{A} : \Gamma \to J(\Gamma)$ is defined by the formula

$$\mathcal{A}(P) = \left( \int_{P_0}^{P} \omega_1, \int_{P_0}^{P} \omega_2, ..., \int_{P_0}^{P} \omega_g \right),$$

where $P \in \Gamma,\ P_0$ is a fixed point on $\Gamma$ and all contours of integration are the same. Considered as a map from the Riemann surface $\Gamma$ to its Jacobian $J(\Gamma)$, the Abel map is well-defined: it does not depend on the choice of the contour.

An integral linear combination of points on $\Gamma$ is called a divisor:

$$D = \sum_{j=1}^{k} n_j P_j,\ P_j \in \Gamma,\ n_j \in \mathbf{Z}.$$

The number $\deg D = \sum_{j=1}^{k} n_j$ is called the degree of the divisor $D$. A divisor corresponding to a meromorphic function on $\Gamma$ is called principal (a zero is given weight 1 and a pole $-1$, multiplicities should be counted). One can extend by linearity the notion of the Abel map to divisors: if $D = \sum_{i=1}^{k} n_j P_j$ then we define

$$\mathcal{A}(D) = \sum_{j=1}^{k} n_j \mathcal{A}(P_j).$$

The famous Abel theorem says that a divisor $D$ is principal if and only if $\mathcal{A}(D) \equiv 0(\text{mod}\Lambda)$ and $\deg D = 0$. The classical Riemann $\theta$-function is defined by its Fourier series:

$$\theta(\mathbf{z}|B) = \sum_{\mathbf{n} \in Z^g} \exp(\frac{1}{2}\langle B\mathbf{n}, \mathbf{n}\rangle + \langle \mathbf{n}, \mathbf{z}\rangle),$$

where $B$ is the matrix of the $b$-periods, $\mathbf{z} = (z_1, ..., z_g) \in \mathbf{C}^g$, $\mathbf{n} = (n_1, ..., n_g) \in \mathbf{Z}^g$ and $\langle\ ,\ \rangle$ is the standard inner product in $\mathbf{C}^g$. The function $\theta(\mathbf{z})$ is entire on $\mathbf{C}^g$. Along the vectors of the lattice the theta function transforms according to the formula:

$$\theta(z + 2\pi i n + B\mathbf{m}) = \exp\left(-\frac{1}{2}\langle B\mathbf{m},\ \mathbf{m}\rangle - \langle \mathbf{m}, z\rangle\right) \cdot \theta(z),$$

which means that $\theta(z)$ is not defined on the Jacobian $J(\Gamma)$.

In the next sections we shall need the following corollary of the Riemann theorem about the zeroes of the theta function.

**Proposition 2.1.1.** Let $F(P) = \theta(\mathcal{A}(P) - \mathcal{A}(D) - K)$, where $P \in \Gamma$, $D = P_1 + ... + P_g$ is a divisor in general position, and $K$ (the vector of Riemann constants) is a vector characteristic for $\Gamma$. Then F(P) has exactly $g$ zeroes: the points of the divisor $D$.

The differential of the second kind $\Omega_A^l$ has its only singularity at the point $A$ on the Riemann surface and in a local parameter $z$ it has the expansion

$$\Omega_A^l = d(z^{-l} + \sum_{r=0}^{\infty} b_{lr} z^r).$$

The differential is normalized by the condition that all its $a$-periods are zeroes. It is easy to show that following symmetry relations holds

$$rb_{lr} = lb_{rl}. \tag{2.1.1}$$

An abelian differential of third type is a differential whose only singularities are two single poles $P$ and $Q$ with residues 1 and $-1$ correspondingly. Such a differential is normalized the same way as before. Any meromorphic differential can be written as a linear combination of differentials of all three types.

If the differentials of the first kind have in the same local parameter a form $\omega_j = \phi_j(z)dz$, then the following formulas are valid for the $b$-periods of normalized differentials of second and third types:

$$\begin{aligned}
&\int_{b_j} \Omega^l_A = -\frac{1}{(l-1)!}\frac{d^{l-1}}{dz^{l-1}}\phi_j(z)|_{z=0},\ j = 1,...,g; \\
&\int_{b_j} \Omega_{PQ} = \int_Q^P \omega_j.
\end{aligned} \tag{2.1.2}$$

Now let us consider two differentials $\Omega^l_A$ and $\Omega^{l'}_{A'}$ and the corresponding local parameters $z_A$ and $z_{A'}$. Then we have the expansions:

$$\int_{P_0}^P \Omega^l_A = \sum_0^\infty c_{lr} z^r_{A'},\ P \to A',$$

$$\int_{P_0}^P \Omega^{l'}_{A'} = \sum_0^\infty d_{l'r} z^r_A,\ P \to A.$$

Then the following symmetry relation holds for their coefficients:

$$rc_{lr} = ld_{rl}. \tag{2.1.3}$$

## 2.2. An algebraic-geometrical element of the Grassmannian

In [4] there is an example of elements of **Gr** corresponding to soliton solutions. We want to find an element $W \in \mathbf{Gr}$ for algebraic geometrical solutions of the hierarchy. Let $\Gamma$ be a Riemann surface of genus $g$, realized as an $n$-sheeted covering of the Riemann sphere $\mathbf{CP}^1$. Let $C_i$ , $i = 1, ....m$ be as before small circles on $\mathbf{CP}^1$ surrounding the points $a_i$ and $\Omega$ be the part of $\mathbf{CP}^1$ outside these circles. (We can always arrange that all branching points of $\Gamma$ lie in $\Omega$, and that the infinity is not a branching point.) Let $\Gamma^-$ be the part of $\Gamma$ covering $\Omega$. Now we take a divisor $D$ of $g + n - 1$ points of multiplicities 1 on $\Gamma^-$. As above elements of the Hilbert space $H$ are sets of $m$ vector functions $\{\mathbf{f}_i\}$ on the circles $C_i$, expanded in series of $z - a_i$. The nonnegative and the negative parts of those expansions we write as $\mathbf{f}_i^+$ and $\mathbf{f}_i^-$ correspondingly. Let $\mathbf{f}_i^j$ be the $j$-th component of $\mathbf{f}_i$ and $P_i^j$ be the point on the $j$-th sheet of $\Gamma$ over $z \in \mathbf{CP}^1$ $(j = 1, 2, .., n)$. We define $W$ as a subspace that consists of $\{\mathbf{f}_i\} \in H$, such that $\mathbf{f}_i^j(z) = f(P_i^j)$, where $f(P)$ is a scalar function on $\Gamma^-$, holomorphic everywhere except (possibly) at the points of the divisor $D$, where it may have simple poles. (In other words, $f \in \mathcal{O}_D(\Gamma^-)$.)

**Theorem 2.2.1.** $W$ belongs to the Grassmannian **Gr** .

*Proof.* In order to have $W \in \mathbf{Gr}$ we need to show :

*i*) $(z - a_1)^{-1}W = (z - a_2)^{-1}W = ... = (z - a_n)^{-1}W \subset W$ and

*ii*) index $P^*|_W = 0$.

Clearly $(z - a_i)^{-1}W \subset W$ : $(z - a_i)^{-1}$ is holomorphic in $\Omega$. Also

$$(z - a_i)/(z - a_j)W \subset W$$

since $(z - a_i)/(z - a_j)$ is holomorphic in $\Omega$ (at $\infty$ this expression is equal to 1), therefore $(z - a_i)^{-1}W = (z - a_j)^{-1}W$ for any $i,\ j = 1, ..., m$. Now we need to check ii). As in [5] we have

$$\text{ind } P^-|_W = \dim H_0(\Gamma, \mathcal{O}_D) \ - \ \dim H_1(\Gamma, \mathcal{O}_D).$$

We need to know ind $P^*|_W$. Let us recall how $P^*|_W$ acts:

$$P^*\{\mathbf{f}_i\} = \{\sum_i \mathbf{f}_i^- + \mathbf{c}\},$$

where $\mathbf{c} = m^{-1}\sum_i \mathbf{g}_i(a_i)$, $\mathbf{g}_i = \mathbf{f}_i^+ - \sum_{k\neq i}\mathbf{f}_k^-$. Let $W$ be an element of the Grassmannian. Let

$$I^- = \text{Im } P^-|_W \text{ and } I^* = \text{Im } P^*|_W.$$

An element $\{\mathbf{f}_i\} \in W$ belongs to Ker $P^*|_W$ if it has a form $\{\mathbf{f}_i^+ - \mathbf{c}\}$, where $\mathbf{c} = \mathrm{m}^{-1}\sum_i \mathbf{f}_i^+(a_i)$ , and it belongs to Ker $P^-|_W$ if it has a form $\{\mathbf{f}_i^+\}$. Let $C_n$ be the space of boundary values of vector-functions constant on $\Omega$. Clearly $C_n \subset H^*$ and $C_n \cong \mathbf{C}^n$. Also let $C_k = I^* \cap C_n$ and dim $\mathrm{C}_k = k$. Obviously $1 \leq k \leq n$ (k is at least 1 since $(1,1,...,1) \in C_k$). In order to finish the proof of the Theorem we need two lemmas.

**Lemma 2.2.1.**

$$\text{dim Ker } P^-|_W - \text{dim Ker } P^*|_W = k$$

*Proof.* We have the exact sequence:

$$0 \to \text{Ker } P^*|_W \xrightarrow{i} \text{Ker } P^-|_W \xrightarrow{\delta} C_k \to 0,$$

where $\delta(\{\mathbf{f}_i^+\}) = \mathrm{m}^{-1}\sum_i \mathbf{f}_i^+(a_i)$.

**Lemma 2.2.2.**

$$\text{dim Coker } P^*|_W - \text{dim Coker } P^-|_W = n - k$$

*Proof.* Let $\alpha = P^-|_{H^*}$. It is clear that

$$\text{Ker } \alpha = C_n \text{ and Ker } \alpha|_{I^*} = I^* \cap C_n = C_k.$$

We can write a commutative diagram, where all the sequences are exact :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_k & \xrightarrow{i} & I^* & \xrightarrow{\alpha} & I^- & \to & 0 \\
 & & \downarrow i & & \downarrow i & & \downarrow i & & \\
0 & \to & C_n & \xrightarrow{i} & H^* & \xrightarrow{\alpha} & H^- & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_n/C_k & \xrightarrow{i} & H^*/I^* & \xrightarrow{\alpha} & H^-/I^- & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The horizontal arrows in the last line are induced by the horizontal arrows in the two top lines and by all the vertical arrows as usual.

This implies that

$$H^-/I^- \cong (H^*/I^*)/(C_n/C_k)$$

and

$$\dim\, H^-/I^- = \dim\, H^*/I^* - (n-k)$$

which is what we needed to show. The lemma is proved.

Now we can compute the index of $P^*|_W$ :

$$\begin{aligned}
\text{ind}\; P^*|_W &= (\dim\, \mathrm{Ker}(P^-|_W) - k) - (\dim\, \mathrm{Coker}(P^-|_W) + n - k) = \\
&= \dim\, \mathrm{Ker}(P^-|_W) - \dim\, \mathrm{Coker}(P^-|_W) - n = \\
&= \dim\, H_0(\Gamma, \mathcal{O}_D) - \dim\, H_1(\Gamma, \mathcal{O}_D) - n = \\
&= \deg\, D - g + 1 - n
\end{aligned}$$

The last step is obtained after using Riemann-Roch theorem. We chose $D$ such that $\deg\, D = g+n-1$, therefore from the last formula $\text{ind}\; P^*|_W = 0$. (Recall the statement of the Riemann-Roch theorem [14]: Let $\Gamma$ be a Riemann surface of genus $g$ and $D$ be a divisor on $\Gamma$. Then

$$\dim\, H_0(\Gamma, \mathcal{O}_D) - \dim\, H_1(\Gamma, \mathcal{O}_D) = \deg\, D - g + 1,$$

where $\mathcal{O}_D$ is the sheaf of meromorfic functions on $\Gamma$ submitted to $-D$.) The proof of the Theorem is completed.

**Remark.** It is interesting that the degree of the divisor $D$ does not depend on the number of essential singularities $m$.

## 2.3. Baker functions for the algebraic-geometrical case

Using the results of the previous section we construct a Baker function for the hierarchy. (The Baker-Akhiezer function in [12] is not a Baker function in the sense of section II, it is only gauge-equivalent to a Baker function for **GZS**.)

Let us recall that the elements of the Grassmannian are sets of $m$ row vectors, whose components are the boundary values on the circles $C_{q\gamma}$ of a function on $\Gamma$ ($C_{q\gamma}$ is the circle on the $\gamma$-th sheet of $\Gamma$ over the circle $C_q$ on the Riemann sphere, that surrounds the point $a_q$). The Baker function is a set of $m$ matrices of size $n \times n$.

Let $P_{q\gamma}$ be the point on the $\gamma$- th sheet of $\Gamma$ over the point $a_q$ on the Riemann sphere. Also let $P_\gamma$, $\gamma = 1, .., n$ be the point $z = \infty$ on the $\gamma$-th sheet of the Riemann surface. We will use the notation $\Omega^l_{q\gamma}$ for $\Omega^l_{P_{q\gamma}}$. We will need the following expansions for this integral:

$$\int_{P_0}^{P} \Omega^l_{q\gamma} = (z-a_k)^{-l}\delta_{\gamma\beta}\delta_{qk} + \sum_{r=0}^{\infty} b^{l,r}_{q\gamma,k\beta}(z-a_k)^r, \quad P \to P_{k\beta},$$

$$\int_{P_0}^{P} \Omega^l_{q\gamma} = \sum_{r=0}^{\infty} b^{\alpha,r}_{q\gamma l} z^{-r}, \quad P \to P_\alpha.$$

In this notation the symmetry relation for differentials of second type looks like:

$$l' \cdot b^{ll'}_{q\gamma,k\beta} = l \cdot b^{l'l}_{k\beta,q\gamma}.$$

Let $\mathbf{U}_{q\gamma l}$ be the vector of $b$-periods of a differential of second kind:

$$(\mathbf{U}_{q\gamma l})_j = \int_{b_j} \Omega^l_{q\gamma}.$$

Finally, let us choose any $g$ points from the divisor $D$ and denote them as $D_1$. For each $\alpha$ ($\alpha = 1, .., n$) enumerate the rest $n-1$ points of $D$ as

$$Q_1, .., Q_{\alpha-1}, Q_{\alpha+1}, .., Q_n.$$

(The enumeration of these points could be different for different $\alpha$.)

**Theorem 2.3.1** The Baker function corresponding to the element $W$ of the Grassmannian looks like:

$$w_\alpha(t,P) = \exp\left(\sum_{\gamma\neq\alpha}\int_{P_\alpha}^{P}\Omega_{P_\gamma Q_\gamma}\right)\cdot\exp\left(\sum_{q=1}^{m}\sum_{\gamma=1}^{n}\sum_{l=1}^{\infty}t_{q\gamma l}\Big(\int_{P_0}^{P}\Omega_{q\gamma}^{l}-b_{q\gamma l}^{\alpha,0}\Big)\right)$$
$$\cdot\frac{\theta(\mathcal{A}(P)+\sum t_{q\gamma l}\mathbf{U}_{q\gamma l}+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})}{\theta(\mathcal{A}(P_\alpha)+\sum t_{q\gamma l}\mathbf{U}_{q\gamma l}+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})}$$
$$\cdot\frac{\theta(\mathcal{A}(P_\alpha)-\mathcal{A}(D_1)-\mathbf{K})}{\theta(\mathcal{A}(P)-\mathcal{A}(D_1)-\mathbf{K})},$$

where $\alpha = 1,..,n$, $P_0$ is a fixed (but arbitrary) point on $\Gamma$, and $\mathbf{K}$ is the vector of Riemann constants.

**Remark.** The above formula is almost identical to the one for the AKNS hierarchy in [5]. The motivation for a normalization of $w_\alpha$ at infinity here is however different: in the case of AKNS $w_\alpha$ has essential singularities at infinities, for GZS the essential singularities are at $P_{k\beta}$ and we normalize at infinity to ensure that $w_\alpha$ is a Baker function for the hierarchy.

*Proof.* First we need to show that $w_\alpha(t,P)$ is a single-valued function on $\Gamma$. Let us choose a path connecting $P_0$ and $P$. If we consider another path connecting these two points, we need to add to the integrals involved in the expression for $w_\alpha$ their periods with respect to some cycle. We can represent this cycle as an integral linear combination $\sum_{i=1}^{g} n_i a_i + \sum_{j=1}^{g} m_j b_j$ of the basis contours. Thus by changing the path of integration we have:

$$\int_{P_0}^{P}\Omega_{q\gamma}^{l}\to\int_{P_0}^{P}\Omega_{q\gamma}^{l}+\langle\mathbf{m},\ \mathbf{U}_{q\gamma l}\rangle,$$
$$\mathcal{A}(P)\to\mathcal{A}(P)+2\pi i n+B\mathbf{m},$$
$$\int_{P_\alpha}^{P}\Omega_{P_\gamma Q_\gamma}\to\int_{P_\alpha}^{P}\Omega_{P_\gamma Q_\gamma}+\langle\mathbf{m},\ \mathcal{A}(P_\gamma)-\mathcal{A}(Q_\gamma)\rangle,$$

where $\mathbf{m} = (m_1,..,m_g)$. Let us recall the transformation formula for the Riemann theta function:

$$\theta(z + 2\pi i n + B\mathbf{m}) = \exp\left(-\frac{1}{2}\langle B\mathbf{m},\ \mathbf{m}\rangle - \langle \mathbf{m}, z\rangle\right) \cdot \theta(z).$$

Thus after changing the path connecting $P_0$ and $P$, the function $w_\alpha$ will change by the following factor:

$$\begin{aligned}
&\exp\Big(\sum_{\gamma\neq\alpha}\langle \mathbf{m},\ \mathcal{A}(P_\gamma) - \mathcal{A}(Q_\gamma)\rangle\Big) \cdot \exp\left(\langle \mathbf{m},\ \sum t_{q\gamma l}\mathbf{U}_{q\gamma l}\rangle\right) \\
&\cdot \frac{\exp\left(-\frac{1}{2}\langle B\mathbf{m},\ \mathbf{m}\rangle - \langle \mathbf{m},\ \mathcal{A}(P) + \sum t_{q\gamma l}\mathbf{U}_{q\gamma l} + \sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma) - \mathcal{A}(D) - \mathbf{K})\rangle\right)}{\exp\Big(-\frac{1}{2}\langle B\mathbf{m},\ \mathbf{m}\rangle - \langle \mathbf{m},\ \mathcal{A}(P) - \mathcal{A}(D_1) - \mathbf{K}\rangle\Big)} \\
&= \exp\Big(-\langle \mathbf{m},\ \sum_{\gamma\neq\alpha}\mathcal{A}(Q_\gamma)\rangle\Big) \cdot \exp\Big(\langle \mathbf{m},\ \mathcal{A}(D) - \mathcal{A}(D_1)\rangle\Big) \\
&= \exp\Big(\langle \mathbf{m},\ \mathcal{A}(D) - \mathcal{A}(D_1) - \sum_{\gamma\neq\alpha}\mathcal{A}(Q_\gamma)\rangle\Big) = \exp\left(\langle \mathbf{m},\ \mathbf{0}\rangle\right) = 1,
\end{aligned}$$

which proves that $w_\alpha$ is indeed single-valued.

Next we must show that $w_\alpha$ belongs to $W$ (in other words $w_\alpha$ is holomorphic everywhere on $\Gamma^-$ except (possibly) at the points of $D$, where it may have simple poles). Really, according to the Riemann theorem for the zeroes of the theta function, $w_\alpha$ has simple poles at the points of $D_1$. The rest of its poles come from the first exponent – the poles of the differential $\Omega_{P_\gamma Q_\gamma}$ with residue $-1$, which are exactly the points $Q_\gamma$. Thus $D$ is the divisor of the poles for $w_\alpha$. Clearly $w_\alpha$ is holomorphic elsewhere on $\Gamma^-$.

Next we need to show that in a neighborhood of $P_{k\beta}$ the function $w_\alpha$ has an asymptotic behavior:

$$w_\alpha = \hat{w}_{k\alpha\beta} \cdot \exp\sum_{l=0}^{\infty} t_{k\beta l}(z - a_k)^{-l}, \quad P \to P_{k\beta},$$

where $\hat{w}_{k\alpha\beta}$ is holomorphic in the vicinity of $P_{k\beta}$. Really, let us choose a small neighborhood of $P_{k\beta}$ inside the circle $C_{k\beta}$. Then from the asymptotic behavior of $\int_{P_0}^{P}\Omega^l_{q\gamma}$ around $P_{k\beta}$ and taking into account that $w_\alpha$ has poles only at the points of $D$, the needed asymptotic behavior of $w_\alpha$ follows.

Finally we need to show that:

$$w_\alpha = \delta_{\alpha\beta} + \mathcal{O}(z^{-1}), \quad P \to P_\beta$$

Clearly $w_\alpha$ is holomorphic at the infinite points $P_\beta$. Further $w_\alpha(P_\beta) = \delta_{\alpha\beta}$ (taking into account the zeroes of the first exponent–the poles of differential $\Omega_{P_\gamma Q_\gamma}$, i.e. points $P_\gamma$, $\gamma \neq \alpha$).

Let us denote by $w_{k,\alpha\beta}$ the asymptotic expressions for $w_\alpha$ when $P \to P_{k\beta}$. Then matrices $w_k = \{w_{k,\alpha\beta}\}_{\alpha\beta=1,..,n}$ are boundary values of a function constant at infinity. Thus $\partial_{t_{k'\alpha l}} w_k$ are boundary values of a function vanishing at infinity, which means

$$\{\partial_{t_{k'\alpha l}} w_k\} \in (z - a_j)^{-1} W.$$

Therefore $\{w_k\}$ is a Baker function for the GZS hierarchy. This completes the proof of the theorem.

## 2.4. $\tau$-function in the algebraic-geometrical case

**Theorem 2.4.1.** Functions $\tau_{k,\alpha\beta}(t)$ and $\tau$ exist such that

$$\hat{w}_{k,\alpha\beta}(t,z) = \frac{\tau_{k,\alpha\beta}(..,t_{q\gamma l} - \delta_{kq}\delta_{\beta\gamma}l^{-1}(z-a_k)^l,..)}{\tau(t)} p_{k,\alpha\beta}(z),$$

the function $p_{k,\alpha\beta}(z)$ is holomorphic in a neighborhood of $P_{k\beta}$ and independent of $t$.

The expression for $\tau_{k,\alpha\beta}(t)$ and $\tau$ are given by:

$$\tau_{k,\alpha\beta}(t) = \exp\Big(\sum_{q\gamma l,q',\gamma' l'} \mu_{q\gamma l,q'\gamma' l'} t_{q\gamma l} t_{q'\gamma' l'}\Big) \cdot \exp\Big(\sum_{q\gamma l} \int_{P_\alpha}^{P_{k\beta}} \Omega^l_{q\gamma} t_{q\gamma l}\Big)$$
$$\cdot \frac{\theta(\mathcal{A}(P_{k\beta}) + \sum t_{q\gamma l}\mathbf{U}_{q\gamma l} + \sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma) - \mathcal{A}(D) - \mathbf{K})}{\theta(\mathcal{A}(P_{k\beta}) + \sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma) - \mathcal{A}(D) - \mathbf{K})},$$

$$\tau(t) = \exp\Big(\sum_{q\gamma l,q'\gamma' l'} \mu_{q\gamma l,q'\gamma' l'} t_{q\gamma l} t_{q'\gamma' l'}\Big)$$
$$\cdot \frac{\theta(\sum t_{q\gamma l}\mathbf{U}_{q\gamma l} + \sum_\gamma \mathcal{A}(P_\gamma) - \mathcal{A}(D) - \mathbf{K})}{\theta(\sum_\gamma \mathcal{A}(P_\gamma) - \mathcal{A}(D) - \mathbf{K})},$$

where $\mu_{q\gamma l,q'\gamma' l'} = -\frac{1}{2}l' \cdot b^{ll'}_{q\gamma,q'\gamma'}$.

*Proof.* We can show that $\tau_{k,\alpha\beta}$ is uniquely determined the same way as we did for $w_\alpha$: choose arcs on $\Gamma$ connecting $P_0$ with $P_{k\beta}$ and with $P_\beta$, and carry out all the integrations involved (including $\mathcal{A}(P_{k\beta})$) along these arcs. The integral $\int_{P_\alpha}^{P_{k\beta}}$ is understood as $\int_{P_0}^{P_{k\beta}} - \int_{P_0}^{P_\alpha}$. Then using the usual properties of the differentials $\Omega^l_{q\gamma}$ and $\theta$-functions we see that $\tau_{k,\alpha\beta}$ does not depend on the choice of the arcs. When $\gamma = \beta$ and $q = k$ the following integral is understood to be regularized:

$$\int_{P_\alpha}^{P_{k\beta}} \Omega^l_{k,\beta} = \lim_{P\to P_k^\beta} \int_{P_\alpha}^{P} \left(\Omega^l_{k,\beta} - (z-a_k)^{-l}\right).$$

If we expand the logarithm of the second exponent of expression for $w_\alpha$ around the point $P_{k\beta}$ we have:

$$\sum t_{q\gamma l}\left(\int_{P_0}^{P}\Omega^l_{q\gamma}-b^{\alpha,0}_{q\gamma l}\right)=$$
$$=\sum_{q\gamma l}\sum_{r=1}^{\infty}b^{lr}_{q\gamma,k\beta}t_{q\gamma l}(z-a_k)^r+\sum_{q\gamma l}\left(b^{l0}_{q\gamma,k\beta}-b^{\alpha,0}_{q\gamma l}\right)t_{q\gamma l}+\sum_{l=1}^{\infty}t_{k\beta l}(z-a_k)^{-l}.$$

We want to rewrite the first term of the last expression:

$$\sum_{q\gamma l}\sum_{r=1}^{\infty}b^{lr}_{q\gamma,k\beta}t_{q\gamma l}(z-a_k)^r$$
$$=\sum\mu_{q\gamma l,q'\gamma' l'}(t_{q\gamma l}-\delta_{kq}\delta_{\beta\gamma}l^{-1}(z-a_k)^l)(t_{q'\gamma' l'}-\delta_{kq'}\delta_{\beta\gamma'}l'^{-1}(z-a_k)^{l'})$$
$$-\sum\mu_{q\gamma l,q'\gamma' l'}t_{q\gamma l}t_{q'\gamma' l'}-\sum\mu_{k\beta l,k\beta l'}(ll')^{-1}(z-a_k)^{l+l'}.$$

That can be achieved by setting

$$\mu_{q\gamma l,q'\gamma' l'}=-\frac{1}{2}l'\cdot b^{ll'}_{q\gamma,q'\gamma'},$$

and using the symmetry relation $l'\cdot b^{ll'}_{q\gamma,k\beta}=l\cdot b^{l'l}_{k\beta,q\gamma}$.

We will also need the following expansion:

$$(\mathcal{A}(P)-\mathcal{A}(P_{k\beta}))_j=\int_{P_{k\beta}}^{P}\omega_j=\int_0^{z-a_k}\phi_j(z)dz=\sum_{l=0}^{\infty}\frac{(z-a_k)^{l+1}}{(l+1)!}\phi_j^{(l)}(a_k)$$
$$=-\sum_{l=0}^{\infty}\frac{(z-a_k)^{l+1}}{l+1}\int_{b_j}\Omega^{l+1}_{k,\beta}=-\sum_{l=0}^{\infty}\frac{(z-a_k)^{l+1}}{l+1}(\mathbf{U}_{k\beta(l+1)})_j=$$
$$-\sum_{l=1}^{\infty}\frac{(z-a_k)^l}{l}(\mathbf{U}_{k\beta l})_j.$$

From the above expansion we have

$$\theta(\mathcal{A}(P)+\sum t_{q\gamma l}\mathbf{U}_{q\gamma l}+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})$$

is equal to

$$\theta(\mathcal{A}(P_{k\beta})+\sum(t_{q\gamma l}-\delta_{kq}\delta_{\beta\gamma}l^{-1}(z-a_k)^l)\mathbf{U}_{q\gamma l}+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K}).$$

We need to verify that $\tau_{k,\alpha\beta}$ is indeed a tau-function.

The translation operator $G_{k\beta}(z)f(t) = f(.., t_{q\gamma l} - \delta_{kq}\delta_{\beta\gamma}l^{-1}(z-a_k)^l, ..)$ was defined in chapter I. Now:

$$
\begin{aligned}
&G_{k\beta}(z)(\text{exponential part of } \tau_{k,\alpha\beta})\\
&=\exp\left(\sum \mu_{q\gamma l,q'\gamma' l'}(t_{q\gamma l} - \delta_{kq}\delta_{\beta\gamma}l^{-1}(z-a_k)^l)(t_{q'\gamma' l'} - \delta_{kq'}\delta_{\beta\gamma'}l'^{-1}(z-a_k)^{l'}\right)\\
&\quad\cdot\exp\left(\sum\int_{P_\alpha}^{P_{k\beta}} \Omega^l_{q\gamma}(t_{q\gamma l} - \delta_{kq}\delta_{\beta\gamma}l^{-1}(z-a_k)^l)\right)\\
&=\exp\left(\sum \mu_{q\gamma l,q'\gamma' l'}t_{q\gamma l}t_{q'\gamma' l'} + \sum \mu_{k\beta l,k\beta l'}(ll')^{-1}(z-a_k)^{l+l'}\right)\\
&\quad\cdot\exp\left(-\sum \mu_{q\gamma l,k\beta l'}l'^{-1}t_{q\gamma l}(z-a_k)^{l'} - \sum \mu_{k\beta l,q'\gamma' l'}l^{-1}t_{q'\gamma' l'}(z-a_k)^l\right)\\
&\quad\cdot\exp\left(\sum\int_{P_\alpha}^{P_{k\beta}} \Omega^l_{q\gamma}t_{q\gamma l}\right)\cdot\exp\left(\sum\int_{P_\alpha}^{P_{k\beta}} \Omega^l_{k\beta}l^{-1}(z-a_k)^l\right)\\
&=\exp\left(\sum \mu_{q\gamma l,q'\gamma' l'}t_{q\gamma l}t_{q'\gamma' l'}\right)\\
&\quad\cdot\exp\left(\sum \mu_{k\beta l,k\beta l'}(ll')^{-1}(z-a_k)^{l+l'} + \sum\int_{P_\alpha}^{P_{k\beta}} \Omega^l_{k\beta}l^{-1}(z-a_k)^l\right)\\
&\quad\cdot\exp\left(\sum_{q\gamma l}\sum_{r=1}^{\infty} b^{lr}_{q\gamma,k\beta}t_{q\gamma l}(z-a_k)^r + \sum_{q\gamma l}\left(b^{l0}_{q\gamma,k\beta} - b^{\alpha,0}_{q\gamma l}\right)t_{q\gamma l}\right)
\end{aligned}
$$

Now we notice that the first factor of the last expression will cancel out with the exponent from $\tau(t)$. The second exponent will become a part of $p_{k,\alpha\beta}(z)$. The third exponent can be rewritten as:

$$
\exp\left(\sum_{q=1}^{m}\sum_{\gamma=1}^{n}\sum_{l=1}^{\infty} t_{q\gamma l}\left(\int_{P_0}^{P}\Omega^l_{q\gamma} - b^{\alpha,0}_{q\gamma l}\right)\right)\cdot\exp\left(-\sum_{l=1}^{\infty} t_{k\beta l}(z-a_k)^{-l}\right).
$$

The first exponent above is exactly the second exponent in the expression of the Baker function $w_\alpha$.

Now consider:

$$
\begin{aligned}
&G_{k\beta}(z)(\text{theta part of } \tau_{k,\alpha\beta})\\
=&\frac{\theta(\mathcal{A}(P_{k\beta})+\sum(t_{q\gamma l}-\delta_{kq}\delta_{\beta\gamma}l^{-1}(z-a_k)^l)\mathbf{U}_{q\gamma l}+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})}{\theta(\mathcal{A}(P_{k\beta})+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})}\\
=&\frac{\theta(\mathcal{A}(P_{k\beta})-\sum_{l=1}^{\infty}\frac{1}{l}(z-a_k)^l\mathbf{U}_{k\beta l}+\sum t_{q\gamma l}\mathbf{U}_{q\gamma l}+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})}{\theta(\mathcal{A}(P_{k\beta})+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})}\\
=&\frac{\theta(\mathcal{A}(P)+\sum t_{q\gamma l}\mathbf{U}_{q\gamma l}+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})}{\theta(\mathcal{A}(P_{k\beta})+\sum_{\gamma\neq\alpha}\mathcal{A}(P_\gamma)-\mathcal{A}(D)-\mathbf{K})}.
\end{aligned}
$$

The numerator of the expression above coincides with the numerator of the first fraction in the expression of the Baker function $w_\alpha$. The denominator above will become part of $p_{k,\alpha\beta}(z)$.

We now consider $\tau(t)$. As we noted earlier, the exponent of $\tau(t)$ will cancel out with the first exponent of $G_{k\beta}(z)\tau_{k,\alpha\beta}(t)$. The numerator of $\tau(t)$ coincides with the denominator of the first fraction of the Baker function. The denominator of $\tau(t)$ will be a part of $p_{k,\alpha\beta}(z)$.

Finally the first exponent and the second fraction of the Baker function have poles only in the points of the divisor $D$, which means that their reciprocical functions will be holomorphic in a neighborhood of $P_{k\beta}$ and they will become a part of $p_{k,\alpha\beta}(z)$. The theorem is proved.

## CONCLUDING REMARKS

Although a general definition of a tau function for the multi-pole ZS hierarchy was given and its existence proven [6], we do not have yet a Grassmannian construction for the general tau function. A soliton tau function was constructed in [4]. For the special case of a single pole hierarchy (AKNS) a general Grassmannian definition was given in [8]. In [16] there is an algebraic-geometrical tau function (only for the diagonal elements). The hierarchy they considered is essentially a set of (independent) single-pole hierarchies (without the cross-poles equations).

We filled two more gaps in the Grassmannian theory for the GZS. First, in chapter 2 we gave a Grassmannian definition of the algebraic-geometrical solution of the hierarchy. In chapter 1 we constructed the diagonal tau functions of a general solution to the hierarchy. The multi-pole case turn out to be complex because of the geometric structure of the Hilbert space: the subspace $H^+$ contains nonzero constants. In order to obtain the diagonal tau function we perturbed the mapping $T_W(\mathrm{g})$ in that constant subspace. For the non-diagonal case we obtained an expression containing both diagonal and non-diagonal elements, and they can not be split. We believe that the reason for that was not our inability to find a proper perturbation $R_{k\alpha\beta}$, it seems that $H^+$ is not large enough. We are currently looking at possible extensions of $H^+$.

## REFERENCES


1. Date, E.: *Multi-Soliton Solutions and Quasi-Periodic Solutions of Nonlinear Equations of Sine-Gordon Type*, Osaka J. Math. **19** (1982) 125.

2. Date, E., Jimbo, M., Kashiwara, M., and Miwa, T.: *Transformation groups for soliton equations*, in: Jimbo and Miwa (ed.) Non-linear integrable systems – classical theory and quantum theory, Proc RIMS symposium, Singapore, 1983.

3. Date, E., Jimbo, M., Kashiwara, M., and Miwa, T.: *Operator approach to the Kadomtsev-Petviashvili equation – transformation groups for soliton equations III*, Journ. Phys. Soc. Japan, **50**, 3806-3812, 1981.

4. Dickey, L.A.: *Why the general Zakharov-Shabat equations form a hierarchy*, Com. Math. Phys., **163**, 509-521, 1994.

5. Dickey, L.A.: *On the $\tau$-function of matrix hierarchies of integrable equations*, Journal Math. Physics, **32**, 2996-3002, 1991.

6. Dickey, L.A.: *On $\tau$-functions of Zakharov-Shabat and other matrix hierarchies*, in Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman, Progress in Nonlinear Differential Equations, vol. **26**, 49-73, 1996.

8. Dickey, L.A.: *On Segal-Wilson's definition of the $\tau$-function and hierarchies AKNS-D and mcKP*, in: Integrable systems, The Verdier Memorial Conference, Progress in Mathematics **115**, Birkhäuser, 147-162, 1993.

9. Dickey, L.A.: *Soliton Equations and Hamiltonian Systems*, Advanced series in Mathematical Physics **12**, World Scientific, 1991.

10. Dubrovin, B.A.: *Riemann surfaces and nonlinear equations I*, MGU, Moscow 1986 (Russian).

11. Dubrovin, B.A.: *Theta functions and nonlinear equations*, Russian Math. Surveys, **36**:2 (1981) 11-92.

12. Dubrovin, B.A.,Krichever, I.M., and Novikov, S.P.: Encyclopaedia of Mathematical Sciences. Dynamical Systems IV. "*Integrable Systems I*". (Arnol'd, V.I. and Novikov, S.P. eds), 1990

13. Dubrovin, B.A., Natanzon, S.M.: *Real two-zone solutions of the sine-Gordon equation*, Func. An. Appl. **16** (1982) 21-33.

14. Forster, O.: *Riemannische Flächen*, Springer 1977.

15. Harnad, J., Wisse, M.-A.: *Isospectral flow in loop algebras and quasiperiodic solutions of the sine-Gordon equation*, J. Math. Physics. **34** (1993) 3518-3526.

16. Jimbo, M., Miwa, T.: *Monodromy preserving deformation of linear differential equations with rational coefficients. II*, Physica 2D (1981) 407-448.

17. Jimbo, M., Miwa, T.: *Monodromy preserving deformation of linear differential equations with rational coefficients. III*, Physica 4D (1981) 26-46.

18. Sato, M.: *Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds*, RIMS Kokyuroku, **439**, 30-46, 1981.

19. Segal, G., Wilson, G.: *Loop groups and equations of KdV type*, Publ. Mathem. IHES, **63**, 1-64, 1985.
20. Ueno, K., Takasaki, K.: *Toda lattice hierarchy*, in: Advanced Studies in Pure Mathematics, **4**, World Scientific, 1-95, 1984.
21. Vasilev, S.H.: *Tau functions of algebraic geometrical solutions to the general Zakharov-Shabat hierarchy*, Modern Physics Letters A, vol. **11**, No. 26, 2139-2151, 1996.
21. Zakharov, V.E., and Shabat, A.B.: *Integration of non-linear equations of mathematical physics by the method of inverse scattering*, Fuct. Anal. Appl.,**13**, No3, 13-22 (1979)